Session 16

Recall ...

The Receiver Operating Characteristic (ROC)

The performance of a binary test is characterized by the pair (α, β)

For a likelihood ratio test, we achieve different pairs $(\alpha(k), \beta(k))$ for different threshold values k See next page

The locus of points $\{(\alpha(k), \beta(k));\ k \in (0, \infty)\}$ specifies all achievable (α, β) that can be obtained by varying the threshold k.

Such a curve is called a *Receiver Operating Characteristic* (ROC)

Recall... 16.2

The Likelihood Ratio Test

We can rewrite the Neyman-Pearson decision rule in terms of the *Likelihood Ratio*

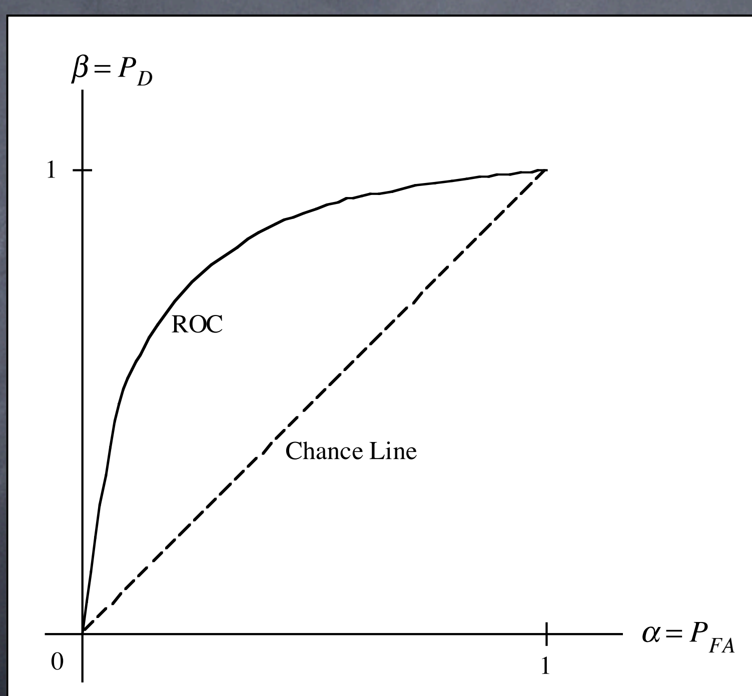
$$L(\underline{x}) = \frac{f_{\underline{\theta}_1}(\underline{x})}{f_{\underline{\theta}_0}(\underline{x})}. \underset{H_0}{\overset{H_1}{\gtrless}} k$$

The Neyman-Pearson test can be rewritten as

$$\phi(\underline{X}) = \begin{cases} 1, & \text{for } L(\underline{x}) > k, \\ \gamma, & \text{for for } L(\underline{x}) = k, \\ 0, & \text{for } L(\underline{x}) < k. \end{cases}$$

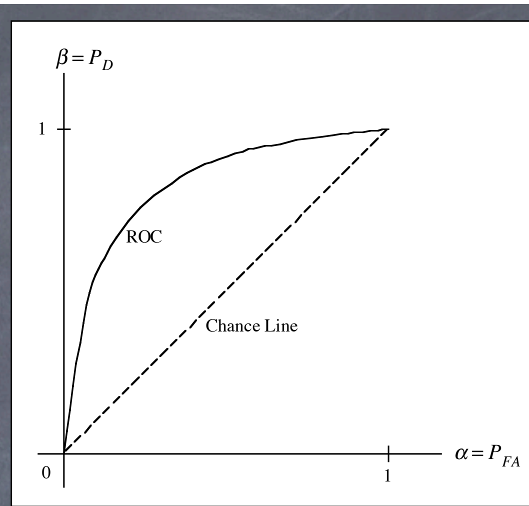
16.3

A typical ROC appears as follows:



16.4

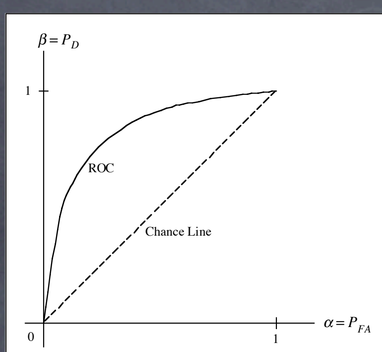
Properties of the ROCs of Likelihood Ratio Tests



1. All continuous likelihood ratio tests have ROCs that are convex downward.

2. Points on the *chance line* ($\beta = \alpha$) can be achieved without observing any data by picking hypothesis H_1 at random with probability α. (i.e., flipping a biased coin with probability α of coming up "heads").

16.5



Properties of the ROCs of Likelihood Ratio Tests (Continued)

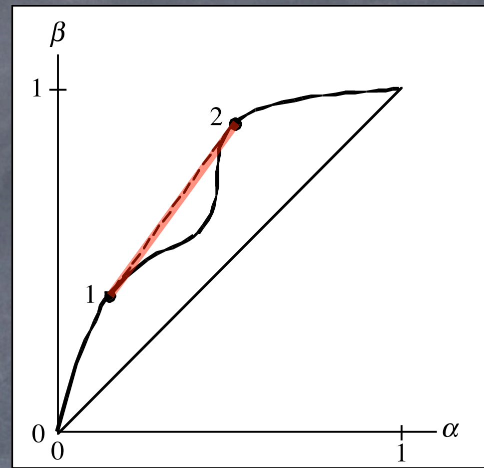
3. All continuous likelihood ratio tests have ROC's that are above the chance line. This is a consequence of property 1, because the points $(\alpha, \beta) = (0, 0)$ and $(\alpha, \beta) = (1, 1)$ are contained on all ROC's.

4. The slope of the ROC at any particular point is given by the threshold achieving that operating point, i.e.,

$$\frac{d\beta}{d\alpha} = \frac{d\beta/dk}{d\alpha/dk} = \frac{f_{L,\theta_1}(\ell)}{f_{L,\theta_0}(\ell)} = k \geq 0$$

Proof of Property 1

Suppose that the ROC of a LRT is non-concave as shown.



Threshold k_1 achieves point 1

Threshold k_2 achieves point 2

A randomized test that selects k_1 with probability p and and k_2 with probability $1 - p$ can achieve any point on the line connecting *point 1* and *point 2*.

But this line is above the LRT's ROC—contradicting the optimality of the LRT. Hence the ROC must be concave downward.

Testing Composite Hypotheses

Recall that when H_0 is composite

$$\alpha = \sup_{\underline{\theta} \in \Theta_0} \left\{ \mathrm{E}_{\underline{\theta}}[\phi(\underline{X})] \right\}$$

and when H_1 is composite

$$\beta(\underline{\theta}) = \mathrm{E}_{\underline{\theta}}[\phi(\underline{X})], \quad \underline{\theta} \in \Theta_1$$

Ideally, we would like to be able to design a test $\phi(\underline{X})$ of size α with the largest possible $\beta(\underline{\theta})$ for each pair $(\underline{\theta}_0, \underline{\theta}_1) \in \Theta_0 \times \Theta_1$.

16.8

Uniformly Most Powerful Tests

Definition: A test $\phi(\underline{X})$ of H_0 versus H_1 is *uniformly most powerful* of size α if it has size α and its power $\beta(\underline{\theta}_1)$ is uniformly greater than the power of any test $\phi'(\underline{X})$ with size $\alpha' \leq \alpha$ testing Θ_0 versus $\{\underline{\theta}_1\}$ for any $\underline{\theta}_1 \in \Theta_1$.

Note that in this definition, the test $\phi'(\underline{X})$ must have size $\alpha' \leq \alpha$ (typically $\alpha' = \alpha$).

Furthermore, $\phi(\underline{X})$ has power $\beta \geq \beta'$.

$\phi'(\underline{X})$ is designed assuming knowledge of the particular $\underline{\theta}_1$ in effect, whereas $\phi(\underline{X})$ is designed only with the knowledge that $\underline{\theta}_1 \in \Theta_1$.

16.9

UMP Tests (Continued)

A UMP test cannot be a function of the actual $\underline{\theta}_1 \in \Theta_1$ in effect.

A UMP test must perform as well as any test designed with knowledge of $\underline{\theta}_1$.

When you ask for a UMP test, you are asking for a lot.

In many detection problems, UMP tests do not exist.

In some detection problems, UMP tests do exist.