Session 12

Narrowband Complex Baseband Noise

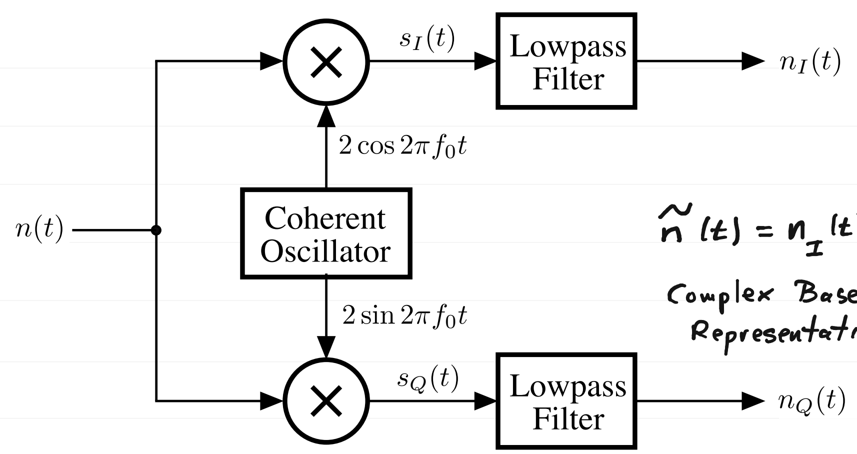
12.1

Consider a real, wide-sense stationary random process $n(t)$.

Assume that $n(t)$ is wide-sense stationary white Gaussian noise.

If we beat $n(t)$ down to baseband, we get a corresponding complex baseband waveform

$$\tilde{n}(t) = n_I(t) + i\, n_Q(t).$$

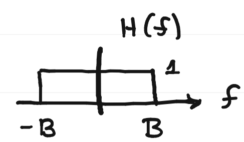


$\tilde{n}(t) = n_I(t) + i\, n_Q(t)$

Complex Baseband Representation

If we have ideal low-pass filters

$$H(f) = \mathbb{1}_{[-B,B]}(f),$$



then

$$S_{n_I n_I}(f) = S_{n_Q n_Q}(f) = \frac{N_0}{2}\, \mathbb{1}_{[-B,B]}(f).$$

Recall...

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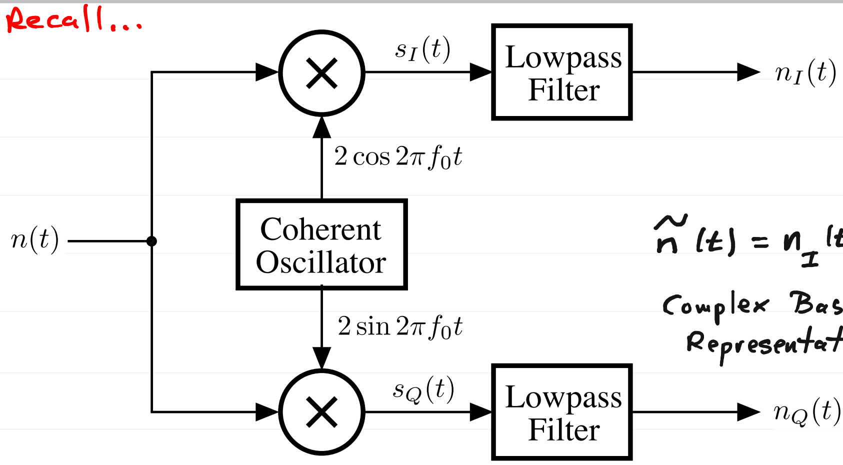
Recall...

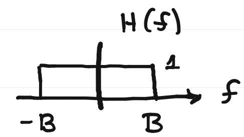


$\tilde{n}(t) = n_I(t) + i\, n_Q(t)$

Complex Baseband Representation

If we have ideal low-pass filters

$$H(f) = 1_{[-B,B]}(f),$$



then

$$S_{n_I n_I}(f) = S_{n_Q n_Q}(f) = \frac{N_0}{2} 1_{[-B,B]}(f).$$

Recall...

Furthermore, $n_I(t)$ and $n_Q(t)$ have identical autocorrelation functions:

$$R_{n_I}(\tau) = R_{n_Q}(\tau) = \int_{-B}^{B} \frac{N_0}{2} e^{i 2\pi f \tau}\, df$$

$$= N_0 B \left(\frac{\sin 2\pi B\tau}{2\pi B\tau} \right)$$

$$= N_0 B \operatorname{sinc}(2B\tau)$$

where

$$\operatorname{sinc}(x) \triangleq \frac{\sin \pi x}{\pi x}.$$

12.6

Recall...

Furthermore, it can be shown that

$$E[n_I(t+\tau)\,n_Q(t)] = -E[n_Q(t+\tau)\,n_I(t)]$$

$$\Rightarrow \quad R_{n_I n_Q}(\tau) = -R_{n_Q n_I}(\tau)$$

It follows that

$$R_{\tilde{n}}(\tau) = E[\tilde{n}(t+\tau)\,\tilde{n}^*(t)]$$

$$= E[(n_I(t+\tau) + i n_Q(t+\tau))(n_I(t) - i n_Q(t))]$$

$$= R_{n_I}(\tau) + R_{n_Q}(\tau) + i\left(E[n_Q(t+\tau)n_I(t)] - E[n_I(t+\tau)n_Q(t)]\right)$$

$$= 2\left(R_{n_I}(\tau) + R_{n_Q n_I}(\tau)\right).$$

From this, we make the following observations...

12.7

1. Since $R_{\tilde{n}}(0) = E[|\tilde{n}(0)|^2]$ is real,

$$E[n_I(t)\,n_Q(t)] = 0$$

$\Rightarrow n_I(t) \perp n_Q(t)$ for all t (uncorrelated)

$\Rightarrow n_I(t) \perp\!\!\!\perp n_Q(t)$, $\forall t$ if jointly Gaussian (independent)

2. From the above definitions

$$E[|\tilde{n}(t)|^2] = 2E[n^2(t)] = 2E[n_I^2(t)] = 2E[n_Q^2(t)].$$

$$\Rightarrow \quad E[n^2(t)] = E[n_I^2(t)] = E[n_Q^2(t)] = \frac{1}{2}E[|\tilde{n}(t)|^2]$$

12.8

3. If the power spectral density (PSD) of $n(t)$ is symmetric about the carrier frequency f_0, (as it is in this case), then the PSD of $\tilde{n}(t)$ is an even function

$\Rightarrow R_{\tilde{n}}(\tau)$ is real for all τ.

$\Rightarrow R_{n_Q n_I}(\tau) = 0, \ \forall \tau$

$\Rightarrow n_I(t) \perp n_Q(t+\tau), \ \forall t, \forall \tau$

$\Rightarrow n_I(t_1) \perp n_Q(t_2), \ \forall t_1, \forall t_2$

$\Rightarrow n_I(t_1) \perp\!\!\!\perp n_Q(t_2), \ \forall t_1, \forall t_2$ when they are jointly Gaussian

$\Rightarrow n_I(t)$ and $n_Q(t)$ are independent Gaussian random processes in our case.

12.9

So going back to our pulse

$$r(t) = \cos(2\pi f_0(t-\tau)) \cdot \mathbb{1}_{[\tau, T+\tau]}(t) + n(t)$$

$$z(t) = e^{i 2\pi f_0 \tau} \cdot \mathbb{1}_{[\tau, T+\tau]}(t) + \tilde{n}(t).$$

$$= (\cos 2\pi f_0 \tau, \sin 2\pi f_0 \tau) \cdot \mathbb{1}_{[\tau, T+\tau]}(t) + (n_I(t), n_Q(t))$$

$$= \underbrace{\left[\cos(2\pi f_0 \tau) \cdot \mathbb{1}_{(\tau, T+\tau)}(t) + n_I(t)\right]}_{\text{"In-phase" component}}$$

$$+ \, i \underbrace{\left[\sin(2\pi f_0 \tau) \cdot \mathbb{1}_{(\tau, T+\tau)}(t) + n_Q(t)\right]}_{\text{"Quadrature" component}}$$

12.10

For any t, $\tilde{n}(t) = n_I(t) + i\, n_Q(t)$ is a circular Gaussian process:

$$E[n_I(t)] = E[n_Q(t)] = 0$$

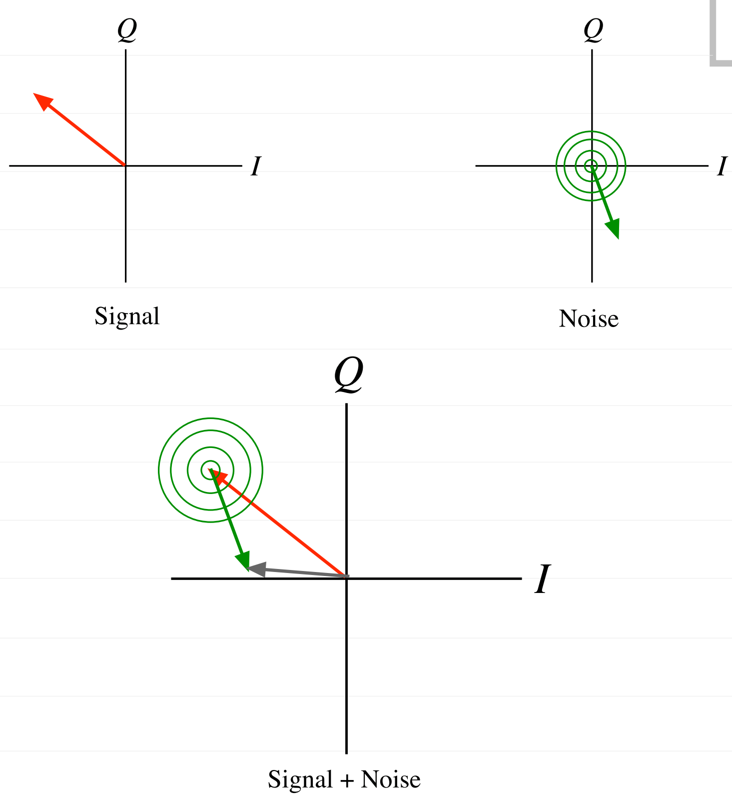
$$\mathrm{var}[n_I(t)] = \mathrm{var}[n_Q(t)] = \sigma^2 = N_0 B.$$

$$n_I(t) \perp\!\!\!\perp n_Q(t)$$

or more generally

$$n_I(t_1) \perp\!\!\!\perp n_Q(t_2), \quad \forall t_1, t_2$$

12.11



12.12

Detection Theory

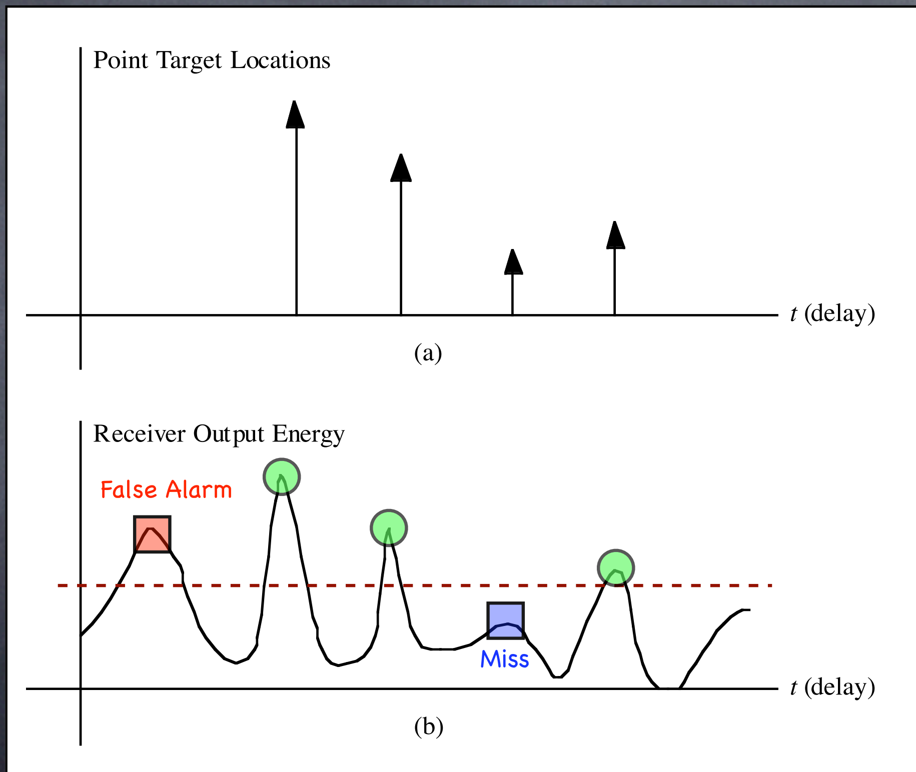
- The most basic task of a RADAR system is to detect whether or not an object (target) is present.
- We may also wish to detect if some "feature" is present in the scattered signal:
  - Spectral Component
  - signal family characteristic
  - depolarized wave component

12.13

- This almost always involves making a decision in the presence of noise.
- Noise can generally—at best—be described statistically (probabilistically.)
- Statistical Decision Theory/Hypothesis Testing is the tool of choice for these problems.
- Engineers usually call this area of statistics Detection Theory.

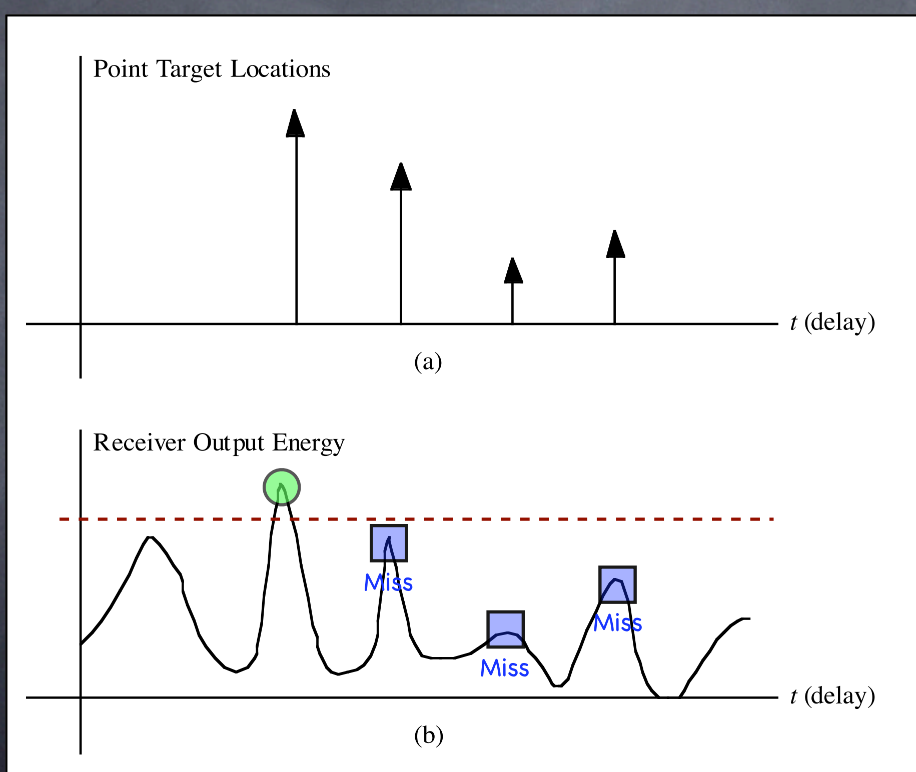
12.14

Example: Detection Based on Energy (A-Scope)



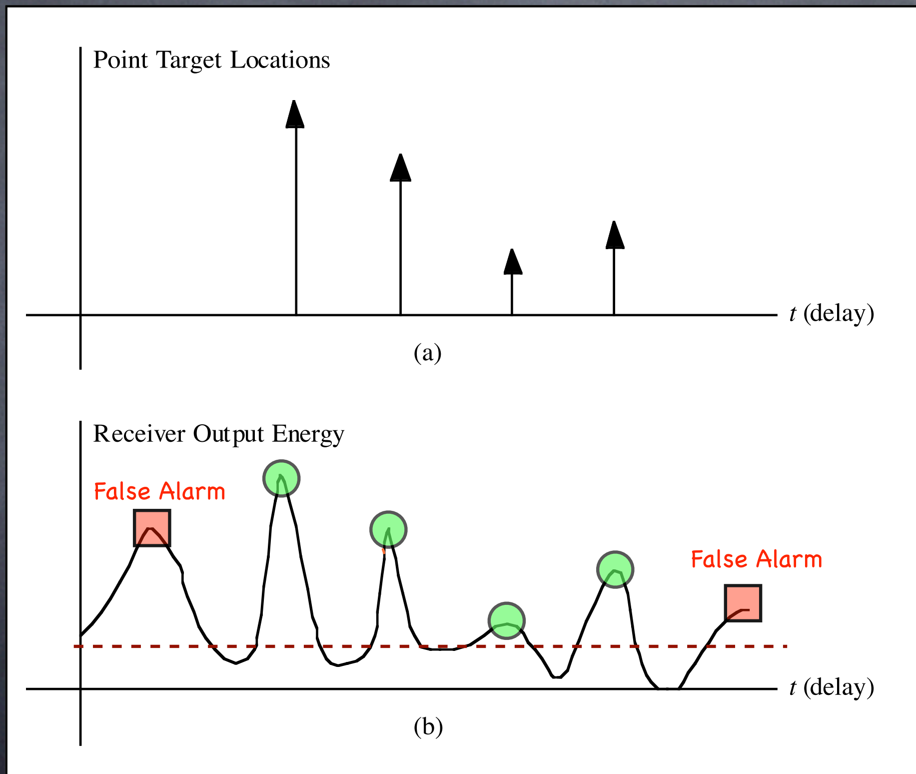
12.15

Don't Like False Alarms: Raise the Threshold



12.16

Don't Like Misses: Lower the Threshold



12.17

An Optimal Test?

- How do we design an optimal test in this case?
- We want as small a number of "false alarms" as possible.
- We want as small a number of "misses" as possible.
- Clearly theses two goals contradict each other.
- An "optimal" test must trade these two goals off against each other. But how?

12.18

In general, we deal with observations from our sensing system of the form

$$y(t) = y[s(t), n(t), \underline{\alpha}(t)]$$

where

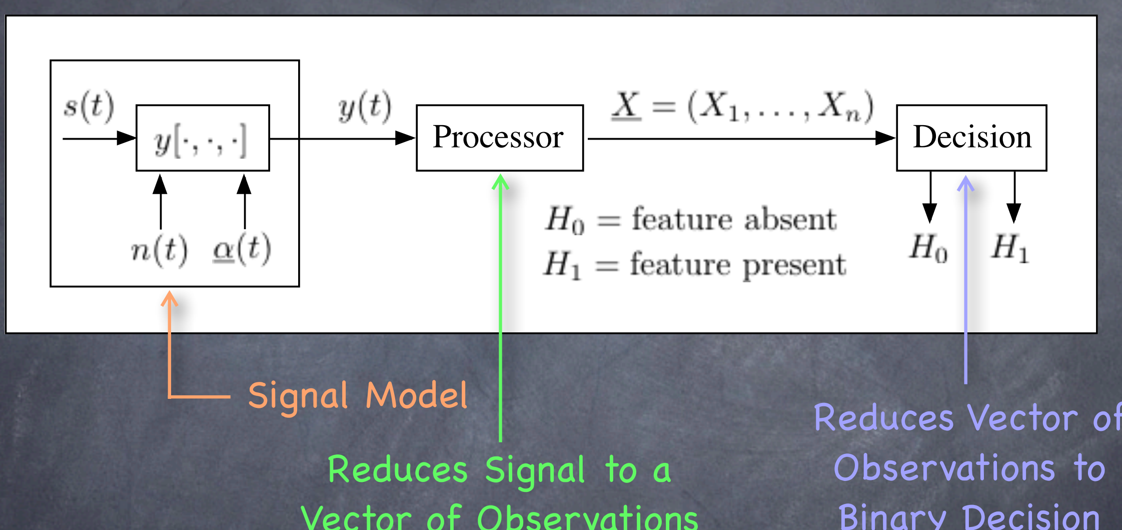
$s(t)$ = transmitted signal

$n(t)$ = noise in received signal

$\underline{\alpha}(t)$ = "state" of the *system* (radar, target, medium)

12.19

General Framework for Radar Detection



Input to processor is a random process $y(t)$.

Processor output $\underline{X} = (X_1, \ldots, X_N)^T$ is a vector of jointly distributed random variables—a *random vector*.