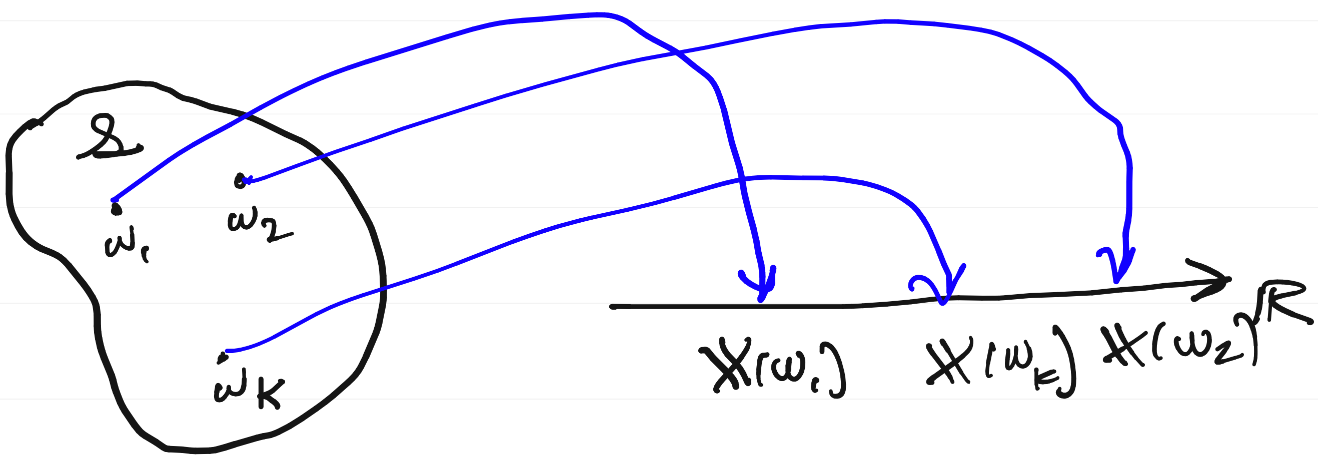
Session 11

Recall ...

11.1

<u>Intuitive "Defn":</u> Given $(\mathcal{S}, \mathcal{F}, P)$, a <u>random variable</u> is a mapping from $\mathcal{S}$ to the real line.

$$X : \mathcal{S} \to \mathbb{R}$$



Recall...

Defn: Given $(\mathcal{S}, \mathcal{F}, P)$ a random variable is a mapping $X: \mathcal{S} \to \mathbb{R}$ with the property that for all $A \in \mathcal{B}(\mathbb{R})$,

$$X^{-1}(A) = \{\omega \in \mathcal{S} : X(\omega) \in A\} \in \mathcal{F}$$

Such a function is called a Borel measurable function.

Recall...

The "Range Space" of a function f

11.3

Let $f : \mathcal{S} \to \mathcal{A}$.

We call the image $f(\mathcal{S}) \subset \mathcal{A}$ the **range space** of f.

e.g. $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$

then "$f(\mathbb{R})$" $= [0, \infty) =$ range space of $f(x) = x^2$.

Given $(\mathcal{S}, \mathcal{F}, P)$ and a R.V X, the R.V. takes on values in the range space

$$\mathcal{C} = f(\mathcal{S}) \subset \mathbb{R}.$$

Because $\mathbb{X}$ is measurable, all subsets of the form

$$\mathbb{X}^{-1}(G) = \{\omega \in \mathcal{S} : \mathbb{X}(\omega) \in G\}$$

where $G \in \mathcal{B}(\mathcal{C})$

must be in the event space $\mathcal{F}$ of $(\mathcal{S}, \mathcal{F}, P)$.

So we can compute the probability $P_X(G)$ that $\mathbb{X}(\omega)$ takes on a value in $G \in \mathcal{B}(\mathcal{C})$.

Here

$$P_X(G) = P(X^{-1}(G))$$

$$= P(\{\omega \in \mathcal{S} : X(\omega) \in G\}), \ \forall G \in \mathcal{B}(\mathcal{C}).$$

So we have the situation

$$(\mathcal{S}, \mathcal{F}, P) \xRightarrow{X(\cdot)} (\mathcal{C}, \mathcal{B}(\mathcal{C}), P_X).$$

So we can focus on the new random experiment $(\mathcal{C}, \mathcal{B}(\mathcal{C}), P_X)$.

<u>Good News:</u> This is a probability space, and we know how to deal with probability spaces.

Q: So why do we describe a RV as a mapping in the first place?

A:

1. Insight
2. We can define multiple RVs on one experiment
3. Generalization to Stochastic Processes.

Examples:

Ex. 1: Let $(\Omega, \mathcal{F}, P)$ be any prob. space. Let $A \in \mathcal{F}$ be any event.

Define $X(\omega) = \mathbb{1}_A(\omega) = \begin{cases} 1, & \omega \in A \\ 0, & \omega \notin A \end{cases}$

Ex. 2 Suppose I toss a die that has six colored sides.

$\Omega = \{Br, R, O, Y, G, Bl\}$

$\mathcal{F} = \mathcal{P}(\Omega)$

Define a RV X as follows:

$$\left.\begin{aligned} X(Br) &= 1 \\ X(R) &= 2 \\ &\vdots \\ X(Bl) &= 6 \end{aligned}\right\}$$

This is a measurable RV.

Ex. 3 Select a resistor at random from a box of resistors and measure its value.

So we can think of a random variable in two ways:

1. <u>Mathematically</u>: A measurable function from $(\Omega, \mathcal{F}, P)$ to $\mathbb{R}$.

2. <u>Intuitively</u>: Something that takes on real values at random.

Given a random variable with range space $\mathcal{C} \subset \mathbb{R}$, we can construct a probability space $(\mathcal{C}, \mathcal{B}(\mathcal{C}), P_X)$.

How do we describe P_X ?

<u>Discrete $\mathcal{C}$</u>: We can use a pmf

$$p_X(x) = P_X(\{x\}), \quad \forall x \in \mathcal{C}.$$

The outcomes of a random experiment $(\mathcal{C}, \mathcal{B}(\mathcal{C}), P_X)$, $\mathcal{C} \subset \mathbb{R}$.

Continuous (uncountable) $\mathcal{C}$: We can use a p.d.f. $f_X(x)$ to assign

$$P_X(G) = \int_G f_X(x)\,dx \quad , \quad \forall G \in \mathcal{B}(\mathcal{C})$$

recall: Since $f_X(x)$ is a pdf:

(i) $f_X(x) \geq 0, \quad \forall x \in \mathcal{C}$

(ii) $\displaystyle\int_{\mathcal{C}} f_X(x)\,dx = 1.$

Defn: Given a random variable $\mathbb{X}$ defined on $(\mathcal{S}, \mathcal{F}, P)$ that induces a new probability space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), P_{\mathbb{X}})$, the *cumulative distribution function* (cdf) of $\mathbb{X}$ is defined as

$$
\begin{aligned}
F_{\mathbb{X}}(\alpha) &\triangleq P_{\mathbb{X}}((-\infty, \alpha]) = P_{\mathbb{X}}(\{x : x \leq \alpha\}) \\
&= P(\{\omega \in \mathcal{S} : \mathbb{X}(\omega) \leq \alpha\}) \\
&= P(\mathbb{X}^{-1}((-\infty, \alpha])), \quad \alpha \in \mathbb{R}.
\end{aligned}
$$

Notation: We often write

$$\text{“}\{ X \le \alpha \}\text{”}$$

or $\text{“}P(\{ X \le \alpha \})\text{”}$

we understand that

$$\text{“}\{ X \le \alpha \}\text{”} = \{ \omega \in \Omega : X(\omega) \le \alpha \} \in \mathcal{F} .$$

$F_X(\alpha)$ specifies $P_X(\cdot)$:

Fact: It is possible to construct any set $F \in B(\mathbb{R})$ using a countable sequence of set operations on the intervals of the form $(-\infty, x_n]$.

n.b. $F_X(x_n) = P_X((-\infty, x_n])$

n.b. We could have defined $B(\mathbb{R})$ as the smallest σ-field containing all intervals of the form $(-\infty, x]$, $x \in \mathbb{R}$

Instead we used the open sets.

11.15

So the cdf is defined as

$$F_X(x) = P(\{X \le x\}), \quad \forall x \in \mathbb{R}$$
$$= P(\{\omega \in S : X(\omega) \le x\}), \quad \forall x \in \mathbb{R}$$
$$= P_X((-\infty, x]), \quad \forall x \in \mathbb{R}.$$

10.23

Properties of the cdf:

1. $F_X(+\infty) = 1$ and $F_X(-\infty) = 0$.

n.b. $F_X(+\infty) = P(\underbrace{\{X \leq +\infty\}}_{S}) \overset{\text{axiom 2}}{=} 1$

$F_X(-\infty) = P(\underbrace{\{X \leq -\infty\}}_{\phi}) = 0.$

2. If $x_1 < x_2$, then
$$F_X(x_1) \le F_X(x_2).$$

n.b. $F_X(x_1) = P_X((-\infty, x_1])$

$F_X(x_2) = P_X((-\infty, x_2])$

$(-\infty, x_1] \subset (-\infty, x_2]$

$\Rightarrow (-\infty, x_2] = (-\infty, x_1] \cup (x_1, x_2]$

disjoint

$$P_X((-\infty, x_2]) = P_X((-\infty, x_1]) + \underbrace{P_X((x_1, x_2])}_{\ge 0}$$

$\Downarrow$ $\Downarrow$

$F_X(x_2) = F_X(x_1)$

$\therefore F_X(x_2) \ge F_X(x_1)$.

3. $P(\{X > \alpha\}) = 1 - F_X(\alpha)$.

$$\{X > \alpha\}^c = \{X \leq \alpha\}$$

$$\Rightarrow \quad P(\{X > \alpha\}) = 1 - P(\{X \leq \alpha\}) = 1 - F_X(\alpha).$$

4. If $x_1 < x_2$, then

$$P(\{x_1 < X \leq x_2\}) = F_X(x_2) - F_X(x_1).$$

n.b. $(-\infty, x_2] = (-\infty, x_1] \cup (x_1, x_2]$

disjoint

$$\Downarrow$$

$$P(\{-\infty < X \leq x_2\}) = P(\{-\infty < X \leq x_1\}) + P(\{x_1 < X \leq x_2\})$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$F_X(x_2) = F_X(x_1) + P(\{x_1 < X \leq x_2\})$$

$$\Rightarrow \quad P(\{x_1 < X \leq x_2\}) = F_X(x_2) - F_X(x_1).$$

5. $P(\{X = x_0\}) = F_X(x_0) - F_X(x_0^-)$

where

$$F_X(x_0^-) = \lim_{\varepsilon \downarrow 0} F_X(x_0 - \varepsilon).$$

n.b. $F_X(x_0) - F_X(x_0^-)$

$$= P_X((-\infty, x_0]) - \lim_{\varepsilon \downarrow 0} P_X((-\infty, x_0 - \varepsilon])$$

$$= \lim_{\varepsilon \downarrow 0} P_X((x_0 - \varepsilon, x_0]) = P_X(\{x_0\})$$

In Summary:

Properties of the cdf:

1. $F_X(+\infty) = 1$ and $F_X(-\infty) = 0$.

2. If $x_1 < x_2$, then $F_X(x_1) \leq F_X(x_2)$.

3. $P(\{X > x\}) = 1 - F_X(x)$, $\forall x \in \mathbb{R}$.

4. If $x_1 < x_2$, then

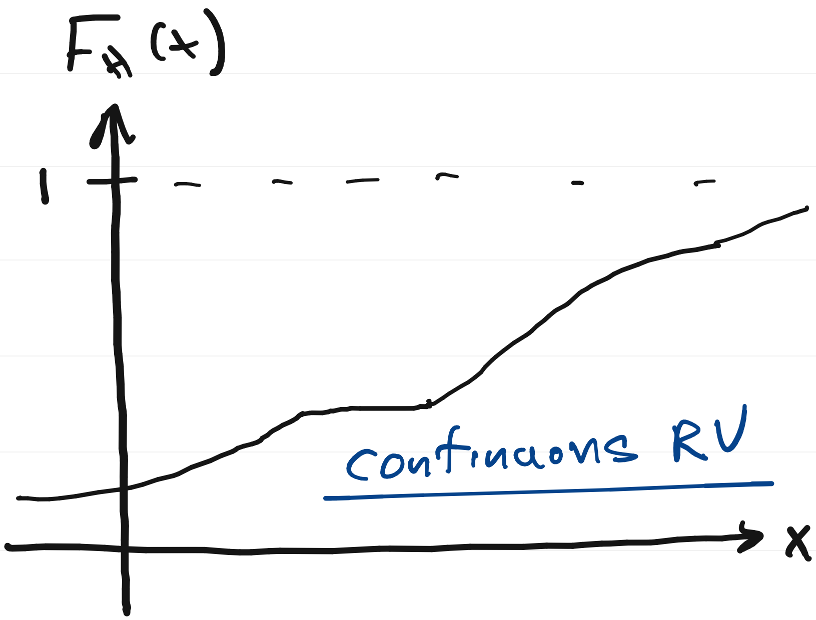
$$P(\{x_1 < X \leq x_2\}) = F_X(x_2) - F_X(x_1).$$

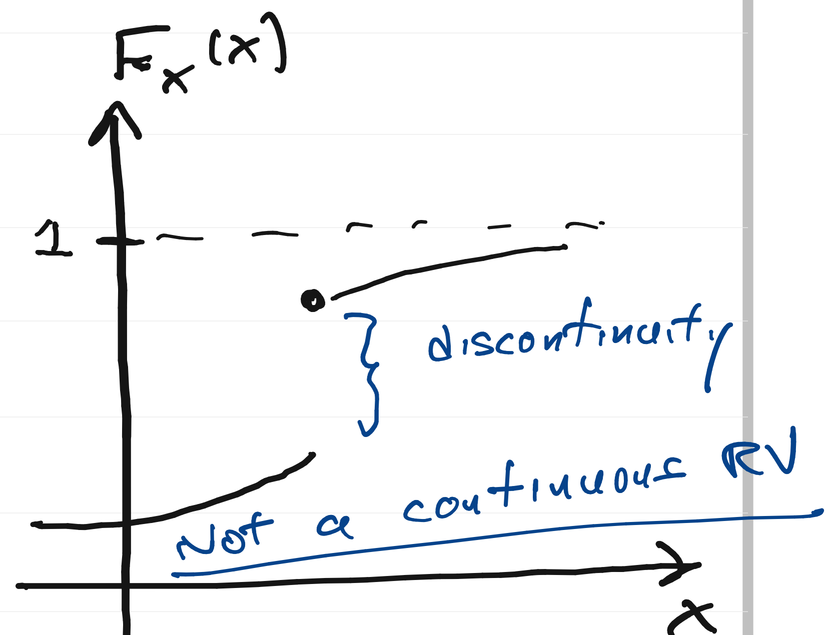
5. $P(\{X = x_0\}) = F_X(x_0) - F_X(x_0^-)$

where

$$F_X(x_0^-) = \lim_{\varepsilon \downarrow 0} F_X(x_0 - \varepsilon)$$

Defn: We say that a random variable is <u>(absolutely) continuous</u> if $F_X(x)$ is a continuous function at <u>all</u> points $x \in \mathbb{R}$.





We say that a RV $\mathbb{X}$ is <u>discrete</u> if it takes on values from a discrete (finite or countable) subset of $\mathbb{R}$.

In this case, $F_{\mathbb{X}}(x)$ is a "<u>staircase function</u>".