Lecture 28

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Rate-Distortion Theory
ECE 642

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Rate-Distortion Theory (Ch. 10 of Cover and Thomas)

- When discussing source coding early on, we emphasized:

  (i) Discrete source alphabets

  (ii) Perfect Reconstruction / Representation

- What about continuous source alphabets?

  - Real numbers—or uncountable subsets of them—require an infinite number of bits to represent perfectly.

  - We are often interested in an approximation requiring only a finite number of bits to represent.

Example: Quantization

- Quantized continuous valued signals are used in many applications:

  (i) Digital Audio

  (ii) Digital Signal Processing

  (iii) Digital Images (digital video and cameras)

  (iv) Digital Radio Receivers (Software Radio)

- When we quantize or approximate continuous valued signals, we introduce error or *distortion*.

- We want ***small distortion*** and ***small rate*** (no. of bits per sample)

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Example: Quantization (Continued)

Question:

- Given a source distribution and a distortion measure, what is the minimum average distortion achievable at a particular rate?

or

- What is the minimum rate required to achieve a particular average distortion?

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Rate-Distortion Theory

- As in Cover and Thomas, we will develop RDT assuming a discrete source, but the results extend easily to continuous alphabets.

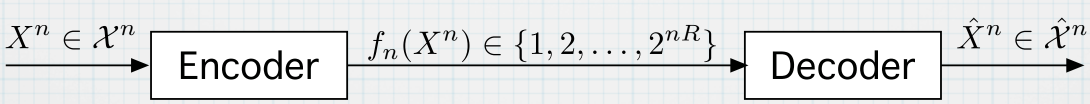
A *source encoder* describes a source sequence

$$X^n = (X_1, \ldots, X_n)$$

from a source with alphabet $\mathcal{X}$ by an index

$$f_n(X^n) \in \{1, 2, \ldots, 2^{nR}\}.$$

The decoder represents X^n by an estimate $\hat{X}^n \in \hat{\mathcal{X}}^n$, where $\hat{\mathcal{X}}$ is the representation alphabet.



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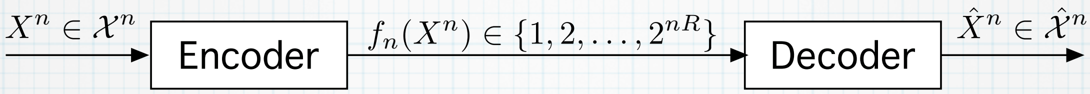

Definition: A **distortion function** or **distance measure** is a mapping

$$d : \mathcal{X} \times \hat{\mathcal{X}} \to \mathbf{R}^+ = [0, \infty)$$

from the set of source-reproduction alphabet pairs to the non-negative reals.

$$d(x, \hat{x}) = \text{distortion in representing } x \in \mathcal{X} \text{ by } \hat{x} \in \hat{\mathcal{X}}$$

Definition: A distortion measure is **bounded** if its maximum value is finite:

$$d_{\max} = \max_{x \in \mathcal{X}, \hat{x} \in \hat{\mathcal{X}}} \{d(x, \hat{x})\} < \infty.$$

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Examples:

1. *Hamming Distortion:* Assume $\mathcal{X} = \hat{\mathcal{X}}$.

$$d(x, \hat{x}) = \begin{cases} 0, & , \quad x = \hat{x}, \\ 1, & , \quad x \neq \hat{x}. \end{cases}$$

2. *Squared-Error Distortion:* Assume $\mathcal{X} = \hat{\mathcal{X}} = \mathbf{R}$.

$$d(x, \hat{x}) = (x - \hat{x})^2.$$

Note that

$$\begin{aligned} \mathrm{E}\left[d(X, \hat{X})\right] &= \mathrm{E}\left[(X - \hat{X})^2\right] \\ &= \text{mean-square error between } X \text{ and } \hat{X}. \end{aligned}$$

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Distortion Measures Between Blocks of Symbols

Definition: The **distortion** between $x^n \in \mathcal{X}^n$ and $\hat{x}^n \in \hat{\mathcal{X}}^n$ is

$$d(x^n, \hat{x}^n) = \frac{1}{n} \sum_{i=1}^{n} d(x_i, \hat{x}_i),$$

where

$$x^n = (x_1, \ldots, x_n),$$
$$\hat{x}^n = (\hat{x}_1, \ldots, \hat{x}_n),$$

and

$$d(x_i, \hat{x}_i) = \text{single letter distortion between } x_i \text{ and } \hat{x}_i.$$

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Rate-Distortion Code

Definition: A $(2^{Rn}, n)$ **rate distortion code** consists of an encoding function

$$f_n : \mathcal{X}^n \to \{1, 2, \ldots, 2^{Rn}\}$$

and a decoding function

$$g_n : \{1, 2, \ldots, 2^{Rn}\} \to \hat{\mathcal{X}}^n.$$

The distortion of the code is

$$\begin{aligned} D &= \mathrm{E}\left[d(X^n, g_n(f_n(X^n)))\right], \\ &= \sum_{x^n \in \mathcal{X}^n} p(x^n) d(X^n, g_n(f_n(X^n))) \end{aligned}$$

where the expectation is taken with respect to the distribution of X^n.

n.b.,

1. The set of n-tuples

$$X^n(1) = g_n(1), \ldots, X^n(2^{Rn}) = g_n(2^{Rn})$$

   is the codebook of the code.

2. The set of pre-images

$$f_n^{-1}(1), \ldots, f_n^{-1}(2^{Rn}) \in \mathcal{X}^n$$

   are the associated *assignment regions* mapped to the associated codewords.

3. In *source coding*, we are interested in designing good codes for representing X^n by $\hat{X}^n$.

Achieveable Rate-Distortion Pairs

Definition: A **rate-distortion pair** (R, D) is said to be **achievable** if there exists a sequence of $(2^{Rn}, n)$ rate distortion codes (f_n, g_n) with

$$\lim_{n\to\infty} \mathrm{E}\left[d(X^n, g_n(f_n(X^n)))\right] \le D.$$

Definition: The **rate-distortion region** for a source is the closure of the set of all achievable (R, D) pairs. We denote it

$$\Gamma_{\mathrm{RD}} = \mathrm{cl}\left\{(R, D) : (R, D) \text{ is achievable.}\right\}$$

Rate-Distortion and *Distortion-Rate* Functions

Definition: The **rate-distortion function** $R(D)$ is the infinum (greatest lower bound) of rates R such that (R, D) is in the rate distortion region of the source for a given distortion D.

Definition: The **distortion-rate function** $D(R)$ is the infinum (greatest lower bound) of distortions D such that (R, D) is in the rate-distortion region of the source for a given rate R.

Note that $R(D)$ and $D(R)$ are opposite ways of looking at the same problem, and $D(R)$ is just the "inverse" of $R(D)$.

Example: For a Gaussian source with variance σ^2 and square error distortion:

$$R(D) = \frac{1}{2}\log_2\frac{\sigma^2}{D}\cdot 1_{[0,\sigma^2]}(D)$$

We will show this.

has "inverse"

$$D(R) = \sigma^2 2^{-2R}\cdot 1_{[0,\infty)}(R).$$

The *Information* Rate Distortion Function

Definition: The **information rate-distortion function** $R^{(I)}(D)$ for a source X with a distortion measure $d(x, \hat{x})$ is defined as

$$R^{(I)}(D) = \min_{p(\hat{x}|x)\in\mathcal{F}(D)}\left\{I(X,\hat{X})\right\},$$

where

$$\mathcal{F}(D) = \left\{p(\hat{x}|x) : \mathrm{E}[d(X,\hat{X})] \le D\right\}.$$

We can think of $p(\hat{x}|x)$ as specifying a hypothetical channel—a "test channel"—between X and $\hat{X}$.