A frequency coded waveform \( s(t) \) is a signal of the form

\[
s(t) = \sum_{l=0}^{N-1} p(t - lT) e^{-j2\pi\Omega_l t},
\]

where

\[
T = \text{chip duration},
\]

\[
p(t) = 1_{[0,T]}(t) \quad \text{(chip waveform)},
\]

and

\[
\Omega_l = d_l/T, \quad l = 1, 2, \ldots N,
\]

where \( \{d_l\} \) is a permutation of the integers 1, 2, ...N.
Frequency Coding Matrices for Frequency-Coded Signals

The Ambiguity Function of Frequency-Coded Waveforms

The ambiguity function of

\[ s(t) = \sum_{l=0}^{N-1} p(t - lT)e^{-j2\pi \Omega_l t} \]

is

\[ \chi_s(\tau, \nu) = \chi_s^{(1)}(\tau, \nu) + \chi_s^{(2)}(\tau, \nu), \]

where

\[ \chi_s^{(1)}(\tau, \nu) = \sum_{m=0}^{N-1} e^{-j2\pi m \nu T} e^{-j2\pi \Omega_m \tau} \chi_p(\tau, \nu), \]

and

\[ \chi_s^{(2)}(\tau, \nu) = \sum_{m=0}^{N-1} \sum_{n=0, n \neq m}^{N-1} e^{-j\pi(\Omega_m + \Omega_n) \tau} e^{-j\pi(m+n)T} \chi_p(\tau + (m - n)T, \nu + (\Omega_n - \Omega_m)) \]
The sidelobes are given by

\[
\chi_s^{(2)}(\tau, \nu) = \sum_{m=0}^{N-1} \sum_{n=0, n \neq m}^{N-1} e^{-j\pi(\Omega_m + \Omega_n)\tau} e^{-j\pi(m+n)T} \cdot \chi_p(\tau + (m-n)T, \nu + (\Omega_n - \Omega_m))
\]

Large contribution when these equal zero!

\[
\tau = (n - m)T \quad \text{and} \quad \nu = (d_n - d_m)/T
\]

or taking \( T = 1 \) for simplicity...

\[
\tau = n - m \quad \text{and} \quad \nu = d_n - d_m
\]

**Coincident Sidelobe Approximation**

- If we consider only the sidelobe contributions due to the situations where both arguments of the ambiguity function is zero, we want to minimize the number of situations where this occurs.

- We especially want to minimize multiple “hits” for any given delay and Doppler shift.

- While this approach only minimizes an approximation of the ambiguity function sidelobes, it is surprisingly effective.

- It is, in fact, the approach John Costas used in designing Costas sequences.
LFM Chirp Sidelobe Overlay Demo

Costas Sidelobe Overlay Demo
Characteristics of Stepped-Frequency Waveforms

- A wide variety of waveforms with different ambiguity functions can be generated.
- These waveforms can be easily generated and amplified for transmission.
- The ambiguity characteristics of these waveforms can be easily visualized because of their localization in time and frequency.
- Provides a straightforward approach to characterizing “ambiguity state” of a target environment.
- These characteristics make them ideal for adaptive waveform radar.

If we count up the number of sidelobe coincidences for each combination of integer delay-Doppler shifts, we can tabulate the coincidences in an array called the *sidelobe array*. 

The Sidelobe Array
Costas Sequences

Definition: A Costas sequence of length $N$ is an integer frequency firing sequence $\{d_1, \ldots, d_N\}$ (or $\{d_0, \ldots, d_{N-1}\}$) that is a permutation of the integers $1, \ldots, N$ (or $0, \ldots, N-1$) such that the maximum sidelobe height or coincidence number in the sidelobe array is 1 for any nonzero integer delay-Doppler shift.

An Example ...


FIGURE 5.4 Partial ambiguity function of a Costas signal with code sequence $\{4, 7, 1, 6, 5, 2, 3\}$.

FIGURE 5.5 Ambiguity function contour at 0.125 (left) compared with the sidelobe matrix (right).
• A Length 40 Costas Sequence:

![Figure 5.9](image)

**FIGURE 5.9** Ambiguity function of a Costas signal (length \(M = 40\)) at all relevant grid points.


• A Length 40 Costas Sequence:

![Figure 5.10](image)

**FIGURE 5.10** Ambiguity function of a Costas signal (length \(M = 40\)) zoom near the origin.

A Length 40 Costas Sequence:

![Graphs showing ACF and spectrum of a Costas signal](image)

**FIGURE 5.11** ACF (top and middle) and the spectrum (bottom) of a Costas signal (length 40).


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**Pushing Sequences:**

A new class of Frequency-Coded Waveforms for Use in Adaptive Waveform Radar

The Ambiguity Function of Frequency-Coded Waveforms

The ambiguity function of

\[ s(t) = \sum_{l=0}^{N-1} p(t - lT) e^{-j2\pi\Omega_l t} \]

is

\[ \chi_s(\tau, \nu) = \chi_s^{(1)}(\tau, \nu) + \chi_s^{(2)}(\tau, \nu), \]

where

\[ \chi_s^{(1)}(\tau, \nu) = \sum_{m=0}^{N-1} e^{-j2\pi\nu T} e^{-j2\pi\Omega_m \tau} \chi_p(\tau, \nu), \]

and

\[ \chi_s^{(2)}(\tau, \nu) = \sum_{m=0}^{N-1} \sum_{n=0, n \neq m}^{N-1} e^{-j\pi(\Omega_m + \Omega_n) \tau} e^{-j\pi(m+n)T} \chi_p(\tau + (m-n)T, \nu + (\Omega_n - \Omega_m)). \]