Session 8

Exam 1 : Tuesday February 6, 2024

Closed Book, Closed Notes,
no Crib Sheets, no calculators

- On Campus FNY Section :
  Here, during Session 9 at 10:30 - 11:45 am
- On Campus Overflow Section :
  Room ARMS 1028 at 10:30 - 11:45 am
  (TA Brad Fitzgerald will proctor.)
- Distance EPE Section :
  See instructions from Lynn Hegewald
  (Using Brightspace / Examity)

Recall...

Independent Experiments

8.1

Sometimes, the outcomes of the two constituent experiments are unrelated:

$$A \times \mathcal{S}_2 \perp\!\!\!\perp \mathcal{S}_1 \times B, \quad \forall A \in \mathcal{F}_1, \; \forall B \in \mathcal{F}_2$$

$\uparrow$ independent

In this case we say that the two experiments $(\mathcal{S}_1, \mathcal{F}_1, P_1)$ and $(\mathcal{S}_2, \mathcal{F}_2, P_2)$ are

<u>independent experiments.</u>

Recall...

8.2

For independent experiments, we assign the probability $P(\cdot)$ as

$$P(A \times B) = P((A \times \mathcal{S}_2) \cap (\mathcal{S}_1 \times B))$$

$$= P(A \times \mathcal{S}_2) \cdot P(\mathcal{S}_1 \times B)$$

$$= P_1(A) \cdot P_2(B)$$

The axioms of probability fill in the probabilities of events that cannot be written as cartesian products (but can be written as a union of disjoint cartesian products.)

8.3

So for two independent experiments $(\mathcal{S}_1, \mathcal{F}_1, P_1)$ and $(\mathcal{S}_2, \mathcal{F}_2, P_2)$ we have a combined experiment $(\mathcal{S}, \mathcal{F}, P)$ with

$$\mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2$$

$$\mathcal{F} = \sigma(\{A \times B : \forall A \in \mathcal{F}_1 \text{ and } \forall B \in \mathcal{F}_2\})$$

$$P(A \times B) = P_1(A) P_2(B)$$

Axioms of probability fill in everything else.

8.4

<u>Generalization to n independent experiments</u>

- Let $(\mathcal{S}_1, \mathcal{F}_1, P_1), (\mathcal{S}_2, \mathcal{F}_2, P_2), \ldots, (\mathcal{S}_n, \mathcal{F}_n, P_n)$ be n random experiments
- Form the combined experiment $(\mathcal{S}, \mathcal{F}, P)$ with $\mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2 \times \cdots \times \mathcal{S}_n$

$$\mathcal{F} = \sigma(\{A_1 \times A_2 \times \cdots \times A_n : \forall A_1 \in \mathcal{F}_1, \forall A_2 \in \mathcal{F}_2, \ldots \forall A_n \in \mathcal{F}_n\})$$

If the experiments are independent, then

$$P(A_1 \times A_2 \times \cdots \times A_n) = P_1(A_1) \cdot P_2(A_2) \cdots P_n(A_n)$$
$$\forall A_1 \in \mathcal{F}_1, \forall A_2 \in \mathcal{F}_2, \ldots, \forall A_n \in \mathcal{F}_n$$

Then the axioms of probability fill in the rest.

Important Special Case: Bernoulli Trials

- Consider a simple experiment $(\mathcal{S}_0, \mathcal{F}_0, P_0)$.
- Now assume I independently perform this experiment n times:

$$(\mathcal{S}_1, \mathcal{F}_1, P_1), (\mathcal{S}_2, \mathcal{F}_2, P_2), \ldots, (\mathcal{S}_n, \mathcal{F}_n, P_n),$$

where

$$\mathcal{S}_1 = \mathcal{S}_2 = \cdots = \mathcal{S}_n \equiv \mathcal{S}_0$$
$$\mathcal{F}_1 = \mathcal{F}_2 = \cdots = \mathcal{F}_n \equiv \mathcal{F}_0$$
$$P_1(\cdot) = P_2(\cdot) = \cdots = P_n(\cdot) \equiv P_0(\cdot)$$

We want to form the combined experiment of these n independent trials.

Our combined experiment:

$$\mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2 \times \cdots \times \mathcal{S}_n$$

$$\mathcal{F} = \sigma(\{\text{all cylinder sets}\})$$

Now suppose I have an event $A \in \mathcal{F}_0$ having probability $P_0(A) = p$, $0 \le p \le 1$.

When I repeat the experiment n times, I want to know the probability of $B_k \in \mathcal{F}$ defined as

$B_k \triangleq$ A occurs exactly k times in the n repetitions of $(\mathcal{S}_0, \mathcal{F}_0, P_0)$

8.7

Let's write the probability of B_k as

$$P_n(k) \triangleq P(B_k).$$

Theorem: $P_n(k) = \binom{n}{k} p^k (1-p)^{n-k}$, $k = 0, 1, 2, \ldots, n$,

where $p = P_0(A)$.

Proof: See Papoulis

We sketch the proof as follows...

8.8

Idea: There are exactly $\binom{n}{k}$ sequences of outcomes of the simple experiment where A occurs exactly k times and $\bar{A}$ occurs $n-k$ times.

e.g. $n=4$, $k=2$ $\binom{4}{2} = \frac{4!}{2!2!} = \frac{4 \cdot 3}{2!} = 6$

$AA\bar{A}\bar{A}$, $A\bar{A}A\bar{A}$, $\bar{A}AA\bar{A}$.

$\bar{A}\bar{A}AA$, $\bar{A}A\bar{A}A$, $A\bar{A}\bar{A}A$

- Each sequence corresponds to a unique $E_j \in \mathcal{F}$
- Each event E_j has probability

$$P(E_j) = p^k (1-p)^{n-k}$$

- The events E_j are disjoint.

$$\therefore \; p_n(k) = P(B_k) = P(E_1 \cup E_2 \cup \cdots \cup E_{\binom{n}{k}})$$
$$= P(E_1) + P(E_2) + \cdots + P(E_{\binom{n}{k}})$$
$$= \binom{n}{k} p^k (1-p)^{n-k}.$$

Example: A box contains two coins, a fair coin and a "strange" coin.

- The fair coin has a probability $\frac{1}{2}$ of coming up "Heads" when tossed.
- The "strange" coin has a probability $\frac{3}{4}$ of coming up "Heads" when tossed.
- I select a coin at random from the box and flip it 4 times. It comes up "Heads" three times.

Q: What is the probability I selected the "strange" coin?

8.11

$\mathcal{S}_1 = \{F, S\}$

$\mathcal{S}_2 = \{HHHH, HHHT, HHTH, \ldots TTTT\}$

$\mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2$ (I select a coin, and I flip it 4 times).

Let F = fair coin selected

S = strange coin selected

A_3 = 3 "Heads" in 4 tosses

What is $P(S|A_3)$?

8.12

$$P(S|A_3) = \frac{P(S \cap A_3)}{P(A_3)} = \frac{P(A_3|S)\,P(S)}{P(A_3)}$$

Assumption $P(S) = \frac{1}{2}$

$$P(A_3|S) = \binom{4}{3}\left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)^1 = 4\left(\frac{27}{64}\right)\left(\frac{1}{4}\right) = \frac{27}{64}$$

$$P(A_3) = P(A_3 \cap \mathcal{S}) = P(A_3 \cap (S \cup F))$$

$$= P((A_3 \cap S) \cup (A_3 \cap F))$$

disjoint

$$= P(A_3 \cap S) + P(A_3 \cap F)$$

$$= P(A_3|S)P(S) + P(A_3|F)P(F)$$

$$= \frac{27}{64} \cdot \frac{1}{2} + \binom{4}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^1 \cdot \frac{1}{2}$$

$$= \frac{27}{128} + \frac{1}{8} = \frac{43}{128}$$

Thus we have

$$P(S|A_3) = \frac{P(A_3|S)\,P(S)}{P(A_3)}$$

$$= \frac{\left(\frac{27}{64}\right)\cdot\left(\frac{1}{2}\right)}{\frac{43}{128}} = \frac{27}{43} \simeq 0.6279$$

n.b. $P(F|A_3) = \dfrac{P(A_3|F)\,P(F)}{P(A_3)} = \cdots = \dfrac{16}{43}$

$\simeq 0.3721$

$P(\cdot\,|A_3)$ $\quad F = \bar{S}$

$P(\bar{S}|A_3) = 1 - P(S|A_3)$

Classical Probability

In classical probability, we have $(\mathcal{S}, \mathcal{F}, P)$ in which

- $\mathcal{S}$ is finite : $|\mathcal{S}| = n$
- $\mathcal{F} = \mathcal{P}(\mathcal{S})$: $|\mathcal{F}| = 2^n$
- All outcomes are equally likely, which means we have pmf
  $p(\omega) = \frac{1}{n}$, $\forall \omega \in \mathcal{S}$

$$\Rightarrow P(A) = \sum_{\omega \in A} p(\omega) = \frac{|A|}{n} = \frac{|A|}{|\mathcal{S}|}$$

So in classical probability, we have

- $P(\phi) = \dfrac{|\phi|}{|S|} = \dfrac{0}{n} = 0.$
- $P(S) = \dfrac{|S|}{|S|} = \dfrac{n}{n} = 1$
- If $A, B \in \mathcal{F}$ are disjoint (i.e $A \cap B = \phi$) then

$$P(A \cup B) = \frac{|A \cup B|}{|S|} = \frac{|A| + |B|}{|S|}$$

$$= \frac{|A|}{|S|} + \frac{|B|}{S} = P(A) + P(B).$$

What about $A, B \in \mathcal{F}$ that are <u>not</u> disjoint?

$$P(\underline{A \cup B}) = \frac{|A \cup B|}{|S|} = \frac{|A| + |B| - |A \cap B|}{|S|}$$

$$= \frac{|A|}{|S|} + \frac{|B|}{|S|} - \frac{|A \cap B|}{|S|}$$

$$= P(A) + P(B) - P(A \cap B).$$

8.17

This extends to general finite unions:

If $A_1, A_2, \ldots, A_n \in \mathcal{F}$ and they are disjoint, then

$$A_i \cap A_j = \phi, \quad \forall i \neq j$$

Then $$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i|$$

$\Rightarrow$ For classical probability, if $A_1, \ldots, A_n$ are disjoint, then

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \frac{\left|\bigcup_{i=1}^{n} A_i\right|}{|\mathcal{S}|} = \sum_{i=1}^{n} \frac{|A_i|}{|\mathcal{S}|} = \sum_{i=1}^{n} P(A_i)$$

8.18

Note also in classical probability that since

$$|\bar{A}| = |\mathcal{S}| - |A|$$

$$P(\bar{A}) = \frac{|\bar{A}|}{|\mathcal{S}|} = \frac{|\mathcal{S}| - |A|}{|\mathcal{S}|}$$

$$= 1 - \frac{|A|}{|\mathcal{S}|} = 1 - P(A).$$

So the classical probability measure behaves like a general axiomatic prob. measure.

It can be argued that Kolmogorov selected the axioms of probability based on how classical probability behaves.

8.19

Exam 1 cut-off

Exam 1 covers Homework Assignments 1 – 3, Lectures 1 – 8.

8.10