Session 5

Examples of Probability Spaces

5.1

Ex. 1: Let $\mathcal{S}$ be a finite sample space and let $\mathcal{F}(\mathcal{S})$ be the power set of $\mathcal{S}$.

Suppose we have a function

$$p(\omega) : \mathcal{S} \to \mathbb{R} \quad \text{such that}$$

(i) $p(\omega) \geq 0, \ \forall \omega \in \mathcal{S}$

(ii) $\sum_{\omega \in \mathcal{S}} p(\omega) = 1.$

This function is called the probability mass function (pmf).

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5.2

We can use the pmf to specify the prob. measure $P(\cdot)$:

$$P(A) = \sum_{\omega \in A} p(\omega) \quad , \forall A \in \mathcal{F}.$$

This will be a valid prob. measure.

n.b. $p(\omega) = P(\{\omega\})$, $\forall \omega \in \mathcal{S}$.

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5.3

Ex. 2: A uniform pmf

Let $\mathcal{S} = \{\omega_1, \omega_2, \ldots, \omega_n\}$ (finite $\mathcal{S}$)

Let $\mathcal{F} = \mathcal{P}(\mathcal{S})$, $|\mathcal{P}(\mathcal{S})| = 2^n$

pmf: $p(\omega) = \frac{1}{n}$, $\forall \omega \in \mathcal{S}$.

$$P(A_k) = \sum_{\omega \in A_k} p(\omega) = \sum_{\omega \in A_k} \left(\frac{1}{n}\right)$$

$$= \frac{1}{n} \sum_{\omega \in A_k} 1 = \frac{|A_k|}{n}$$

classical probability

$$= \frac{|A_k|}{|\mathcal{S}|} \quad , \forall A_k \in \mathcal{F}$$

Ex. 3 Binomial pmf

$$\mathcal{S} = \{0, 1, 2, \dots, n\}, \quad |\mathcal{S}| = n+1$$
$$\mathcal{F} = \mathcal{P}(\mathcal{S}), \quad |\mathcal{F}| = 2^{n+1}$$

pmf: $p(k) = \binom{n}{k} a^k (1-a)^{n-k}$, $a \in [0,1]$
$k = 0, 1, \dots, n$

where $\binom{n}{k} = \dfrac{n!}{k!\,(n-k)!}$

"n choose k"

$$P(A) = \sum_{k \in A} p(k), \quad \forall A \in \mathcal{F}.$$

Is this $p(k)$ a valid pmf?

(i) clearly, $p(k) = \underbrace{\binom{n}{k}}_{>0} \underbrace{a^k}_{\geq 0} \underbrace{(1-a)^{n-k}}_{\geq 0} \geq 0$.

(ii) Must show that $\sum_{k=0}^{n} \binom{n}{k} a^k (1-a)^{n-k} = 1$

* (exercise)

* **Binomial Theorem:**

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

for any two numbers $a, b \in \mathbb{R}$ (or $a, b \in \mathbb{C}$).

You should memorize the Binomial Theorem.
We will use it many times in the course.

Ex. 4 The Geometric pmf:

$S = \{0, 1, 2, \dots\}$ $(S = \{1, 2, 3 \dots\})$

$\mathcal{F} = \mathcal{P}(S)$

pmf: $p(k) = (1-a)a^k$, $a \in (0,1)$,

$k = 0, 1, 2, \dots$

$$P(A) = \sum_{k \in A} p(k) = \sum_{k \in A} (1-a)a^k, \quad \forall A \in \mathcal{F}.$$

Is this a valid pmf

(i) $p(k) = (1-a)a^k \geq 0$, $k = 0, 1, 2, \dots$

(ii) $P(S) = \sum_{k=0}^{\infty} (1-a)a^k = 1$* (exercise)

* Hint: $\sum_{k=0}^{\infty} a^k = \frac{1}{1-a}$, $|a| < 1$.

Ex. 5: The Poisson pmf:

$S = \{0, 1, 2, \dots\}$

$\mathcal{F} = \mathcal{P}(S)$

pmf: $p(k) = \frac{\lambda^k e^{-\lambda}}{k!}$, $k = 0, 1, 2, \dots$

$\lambda > 0$.

$$P(A) = \sum_{k \in A} p(k), \quad \forall A \in \mathcal{F}.$$

Is this a valid pdf?

(i) $p(k) = \frac{\lambda^k e^{-\lambda}}{k!} \geq 0$, $k = 0, 1, 2, \dots$

(ii) $P(S) = \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} = 1$ *(exercise)

* Hint: $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$

Let's look at an uncountable sample space:

E.g. $\mathcal{S} = \mathbb{R}$

$\mathcal{F} = \mathcal{B}(\mathbb{R})$

$P(\cdot)$ - How do we do this.

We introduce the probability density function (pdf) to assign probabilities $P(A)$, where $A \in \mathcal{B}(\mathbb{R})$.

Properties of the pdf:

A probability density function (pdf) is a function that maps the sample space $\mathcal{S} = \mathbb{R}$,

$$f : \mathcal{S} \to \mathbb{R}$$

satisfying the following properties:

(i) $f(r) \geq 0 \quad , \forall r \in \mathbb{R}$

(ii) $\displaystyle\int_{-\infty}^{\infty} f(r)\, dr = 1.$

Given a valid pdf $f(r)$, we get a valid probability measure $P(\cdot)$ for any $A \in B(\mathbb{R})$ by integrating:

$$P(A) = \int_A f(r)\,dr = \int_{-\infty}^{\infty} f(r) \cdot \mathbb{1}_A(r)\,dr,$$

where $\mathbb{1}_A(r) = \begin{cases} 1, & r \in A \\ 0, & r \notin A \end{cases}$

is called the indicator function of the set A

Q: Does $P(A) = \int_{-\infty}^{\infty} f(r) \cdot \mathbb{1}_A(r)\,dr$

give a valid prob. measure for every $A \in B(\mathbb{R})$?

Consider the Riemann integral

$$\int_{-\infty}^{\infty} f(r) \cdot \mathbb{1}_A(r)\,dr$$

- It will be well defined for any A that is an interval:
  $A = (a,b)$ or $[a,b]$, $a < b$.
- It's also well defined for an A equal to a finite union of intervals.

Example of a set A where the Riemann integral does not exist

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Suppose I have a pdf

$$f(r) = 1_{[0,1]}(r) = \begin{cases} 1, & 0 \le r \le 1 \\ 0, & \text{elsewhere.} \end{cases}$$

Let $A = \mathbb{Q}$ = rational numbers.

$$P(A) = P(\mathbb{Q}) = \int_{-\infty}^{\infty} f(r) \cdot 1_{\mathbb{Q}}(r)\, dr$$

$$= \int_0^1 1_{\mathbb{Q}}(r)\, dr$$

does not exist as a Riemann integral

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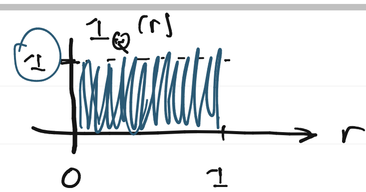
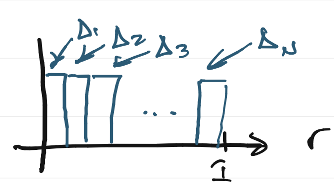
$$P(\mathbb{Q}) = \int_{-\infty}^{\infty} f(r) \cdot 1_{\mathbb{Q}}(r)\, dr$$



$$= \int_0^1 1_{\mathbb{Q}}(r)\, dr$$

$$= \lim_{N\to\infty} \sum_{k=1}^{N} |\Delta_k| \cdot 1_{\mathbb{Q}}(\hat{x}_k)$$

$$\hat{x}_k \in \Delta_k$$



$$\underline{\text{SUM}}_N = \sum_{k=1}^{N} |\Delta_k| \cdot \left(\min_{x\in\Delta_k} 1_{\mathbb{Q}}(x)\right) = \sum_{k=1}^{N} |\Delta_k| \cdot 0 = 0$$

$$\overline{\text{Sum}}_N = \sum_{k=1}^{N} |\Delta_k| \cdot \left(\max_{x\in\Delta_k} 1_{\mathbb{Q}}(x)\right) = \sum_{k=1}^{N} |\Delta_k| \cdot 1 = 1$$

For a Riemann integral to exist

$$\lim_{N\to\infty} \underline{\text{Sum}}_N = \lim_{N\to\infty} \overline{\text{Sum}}_N .$$

$\therefore$ The Riemann integral does not exist in this case.