

Session 30

Homework Assignment 10 Most Important Problems

1 - 7 , 9 , 14 , 15

Also , please fill out course
evaluations !

Defn: The mean of a R.P. $x(t)$ is

30.1

$$\mu_x(t) \triangleq E[x(t)]$$

Defn: The autocorrelation function of a R.P. $x(t)$ is

$$R_{xx}(t_1, t_2) \triangleq E[x(t_1)x(t_2)]$$

n.b. The autocorrelation function is just a measure of the correlation in $x(t)$ at time t_1 and time t_2 .

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n.b. $R_{xx}(t_1, t_2) = E[x(t_1)x(t_2)]$

30.2

$$= E[x(t_2)x(t_1)] = R_{xx}(t_2, t_1)$$

for a real random process.

n.b. The autocorrelation function is symmetric in its time arguments.

For a complex random process $x(t)$

$$\begin{aligned} R_{xx}(t_1, t_2) &= E[x(t_1)x(t_2)^*] = (E[x(t_2)x(t_1)^*])^* \\ &= R_{xx}^*(t_2, t_1) \end{aligned}$$

Conjugate symmetric

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Defn: The autocovariance

30.3

function of a ^{real} random process

$X(t)$ is defined as

$$C_{XX}(t_1, t_2) \triangleq E[(X(t_1) - \mu_X(t_1))(X(t_2) - \mu_X(t_2))]$$

$$(\underbrace{= \dots = R_{XX}(t_1, t_2) - \mu_X(t_1)\mu_X(t_2)})$$

can easily show this

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Defn: A random process $X(t)$
is called a Gaussian random
process if the RVs

30.4

$X(t_1), X(t_2), \dots, X(t_n)$
are jointly Gaussian for
any $n \in \mathbb{N}$ and any set
of sample times $t_1, \dots, t_n$.

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Fact: The n -th order characteristic function of a Gaussian R.P. $X(t)$ is

$$\Phi_{X(t_1), \dots, X(t_n)}(w_1, \dots, w_n) = \exp \left\{ i \sum_{j=1}^n \mu_X(t_j) w_j \right\} \exp \left\{ -\frac{1}{2} \sum_{j=1}^n \sum_{k=1}^n C_{XX}(t_j, t_k) w_j w_k \right\}$$

This is a complete probabilistic description.

Fact: A Gaussian random process $X(t)$ is completely characterized by

- (i) $\mu_X(t) = E[X(t)]$
 and
 (ii) $C_{XX}(t_1, t_2) = E[(X(t_1) - \mu_X(t_1))(X(t_2) - \mu_X(t_2))]$

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Note: If a R.P. $X(t)$ is W.S.S.,
 then

$$\begin{aligned} R_{XX}(t_1, t_2) &= E[X(t_1)X(t_2)] \\ &= f(t_1 - t_2) \\ &= "R_X(t_1 - t_2)" \end{aligned}$$

This is also sometimes written as

$$E[X(t+\tau)X(t)] = R_X(\tau)$$

for a W.S.S. random process.

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Defn: If $X(t)$ is a w.s.s. random process with autocorrelation function $R_X(\tau)$, then the Power Spectral Density of $X(t)$ is defined as

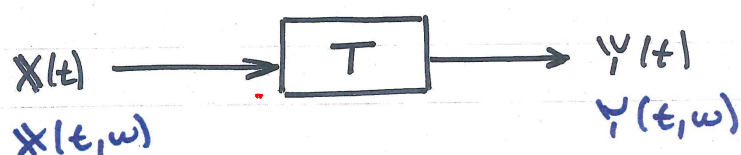
$$S_{XX}(\omega) \triangleq \int_{-\infty}^{\infty} R_X(\tau) e^{-i\omega\tau} d\tau$$

Here ω is radian frequency (not the outcome of experiment.)

$S_{XX}(\omega)$ is a measure of the average distribution of signal power in frequency for the R.P. $X(t)$.

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Systems with Stochastic Inputs



Given a R.P. $X(t)$, if we assign to each sample function $X(t, \omega)$ a new sample function $Y(t, \omega)$, we have a new random process

$$Y(t) = T[X(t)]$$

whose sample functions are

$$Y(t, \omega) = T[X(t, \omega)].$$

n.b. We will assume that

30.9

$T[\cdot]$ is deterministic. (not random)

Think of

$x(t)$ = input to a system

$y(t)$ = output of the system

$$x(t) \rightarrow \boxed{T} \rightarrow y(t)$$

$$x(t, \omega) \rightarrow \boxed{T} \rightarrow y(t, \omega)$$

$$y(t, \omega) = T[x(t, \omega)], \forall \omega \in \Omega.$$

30.10

- We are interested in finding a statistical description of the output $y(t)$ in terms of the statistical description of the input $x(t)$ and the system description $T[\cdot]$
- For general T this is very difficult
- We will look at two special cases:
 1. Linear Time Invariant (L.T.I.) Systems
 2. Memoryless Systems.

Linear Systems

30.11

A linear system $L[\cdot]$ is a transformation rule satisfying the following two properties:

$$1. L[x_1(t) + x_2(t)] = L[x_1(t)] + L[x_2(t)]$$

(Superposition)

$$2. L[A \cdot x(t)] = A \cdot L[x(t)]$$

(Homogeneity)

n.b. A can be an R.V. or a constant

Defn: A (linear) system is time-invariant if, given response $y(t)$ to input $x(t)$ it has response $y(t+c)$ for input $x(t+c)$, for all $c \in \mathbb{R}$.

30.12

A linear time invariant system is characterized by its impulse response $h(t)$:

$$\delta(t) \xrightarrow{S(t-t_0)} \boxed{h(t)} \xrightarrow{h(t-t_0)} h(t)$$

$$x(t) \xrightarrow{\quad} \boxed{h(t)} \xrightarrow{\quad} y(t) = x(t) * h(t).$$

If we put a random process $X(t)$ into a L.T.I. system, we get a random process $Y(t)$ out of the system:

$$\begin{aligned} Y(t) &= X(t) * h(t) = \int_{-\infty}^{\infty} X(t-\alpha) h(\alpha) d\alpha \\ &= \int_{-\infty}^{\infty} X(\alpha) h(t-\alpha) d\alpha \end{aligned}$$

We interpret this on a sample function basis:

$$Y(t, \omega) = X(t, \omega) * h(t), \quad \forall \omega \in \Omega.$$

Important Facts:

1. If the input to a L.T.I. system is a Gaussian R.P., then the output is a Gaussian R.P.
2. If the input to a stable L.T.I. system is S.S.S., so is the output.

An L.T.I. system is stable iff
 $\int_{-\infty}^{\infty} |h(t)| dt < \infty.$ (BIBO stable)

Fundamental Theorem:

For any linear system

$$E[L[X(t)]] = L[E[X(t)]]$$

(This basically reduces to an exchange of orders of integration).

Applying this to a L.T.I. system, we get

$$\begin{aligned} E[Y(t)] &= E\left[\int_{-\infty}^{\infty} X(t-\alpha) h(\alpha) d\alpha\right] \\ &= \int_{-\infty}^{\infty} E[X(t-\alpha)] h(\alpha) d\alpha \\ &= \int_{-\infty}^{\infty} \mu_X(t-\alpha) h(\alpha) d\alpha \\ &= \mu_X(t) * h(t) \end{aligned}$$

$$\therefore \mu_Y(t) = E[Y(t)] = \mu_X(t) * h(t).$$

$$R_{YY}(t_1, t_2) = E[Y(t_1) \cdot Y(t_2)] \quad 30.17$$

$$= E \left[\underbrace{\int_{-\infty}^{\infty} X(t_1 - \alpha) h(\alpha) d\alpha}_{Y(t_1)} \cdot \underbrace{\int_{-\infty}^{\infty} X(t_2 - \beta) h(\beta) d\beta}_{Y(t_2)} \right]$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E[X(t_1 - \alpha) X(t_2 - \beta) h(\alpha) h(\beta) d\alpha d\beta]$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_{XX}(t_1 - \alpha, t_2 - \beta) h(\alpha) h(\beta) d\alpha d\beta$$

If $R_{XX}(t_1, t_2) = R_X(t_1 - t_2)$, then we have

$$R_{YY}(t_1, t_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_X(t_1 - t_2 - \alpha + \beta) h(\alpha) h(\beta) d\alpha d\beta$$

$$= R_Y(t_1 - t_2) \quad (\text{a function of } t_1 - t_2).$$

So if $X(t)$ is a W.S.S. random process, then

30.18

$$(i) \mu_Y(t) = \mu_X * h(t) = \int_{-\infty}^{\infty} \mu_X h(t - \alpha) d\alpha$$

$$= \mu_X \int_{-\infty}^{\infty} h(t - \alpha) d\alpha = \mu_Y$$

$$(ii) R_{YY}(t_1, t_2) = R_Y(t_1 - t_2)$$

Theorem: If the input to a stable L.T.I. System is a W.S.S. random process, then the output is a W.S.S. random process.

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Recall the definition of
 The Power Spectral Density of a W.S.S
 R.P. $X(t)$ is

$$S_{xx}(\omega) \triangleq \int_{-\infty}^{\infty} R_{xx}(\tau) e^{-i\omega\tau} d\tau$$

$\omega = \text{radian/sec.}$

where (where we assume $X(t)$ is W.S.S.)

$$R_{xx}(\tau) = E \left[\underbrace{X(t+\tau)}_{t_1} \underbrace{X^*(t)}_{t_2} \right]$$

for a W.S.S. R.P. $X(t)$

30.20

Note: 1. Because $R_{xx}(-\tau) = R_{xx}^*(\tau)$

$S_{xx}(\omega)$ is a real function

$$2. R_{xx}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xx}(\omega) e^{i\omega\tau} d\omega$$

3. In order to consider $S_{xx}(\omega)$,
 we must assume $X(t)$ is
 W.S.S.

4. Because $R_{xx}(\tau)$ is a non-negative
 definite function, it follows that
 $S_{xx}(\omega) \geq 0$.

Key Result: If $X(t)$ is a W.S.S.

Random Process and it is the input to a stable L.T.I. system with impulse response $h(t)$, then the Power Spectral Density of the output $Y(t)$ is

$$S_{YY}(\omega) = S_{XX}(\omega) |H(\omega)|^2$$

where

$$H(\omega) = \int_{-\infty}^{\infty} h(t) e^{-i\omega t} dt.$$

Defn: A random process $W(t)$ is called a white noise process if

$$C_{WW}(t_1, t_2) = 0, \forall t_1 \neq t_2.$$

Fact: All (non-trivial) W.S.S. white noise processes have

$$R_{WW}(t_1, t_2) = r_0 \cdot \delta(t_1 - t_2)$$

where $r_0 = \text{constant} > 0$.

EXAMPLE: Let $X(t)$ be a W.S.S

30.23

R.P. with PSD $S_{XX}(\omega)$ and let

$Y(t)$ be the "smoothed" random process given by

$$Y(t) = \frac{1}{2T} \int_{t-T}^{t+T} X(\alpha) d\alpha$$

This can be represented by a L.T.I. System with impulse response

$$h(t) = \frac{1}{2T} \cdot 1_{[-T, T]}(t)$$

$$X(t) \rightarrow \boxed{h(t)} \rightarrow Y(t)$$

30.24

$$h(t) = \frac{1}{2T} \cdot 1_{[-T, T]}(t)$$

What is the PSD $S_{YY}(\omega)$ of $Y(t)$?

Solution:

$$S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega)$$

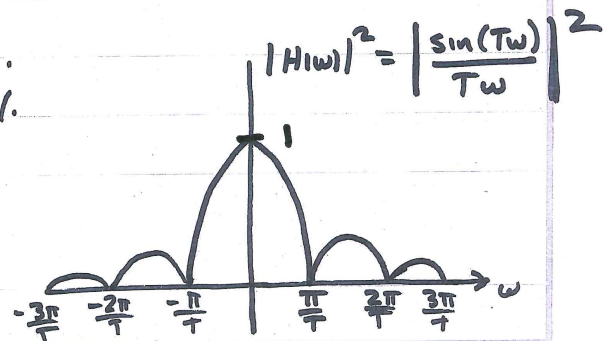
$$H(\omega) = \int_{-\infty}^{\infty} h(t) e^{-i\omega t} dt = \int_{-\infty}^{\infty} \frac{1}{2T} 1_{[-T, T]}(t) e^{-i\omega t} dt$$

$$= \frac{1}{2T} \int_{-T}^T e^{-i\omega t} dt = \frac{1}{2T} \left. \frac{e^{-i\omega t}}{-i\omega} \right|_{-T}^T$$

$$= \frac{e^{-i\omega T} - e^{+i\omega T}}{i2T\omega} = \frac{1}{T\omega} \left[\frac{e^{i\omega T} - e^{-i\omega T}}{i2} \right] = \frac{\sin(T\omega)}{T\omega}$$

$$S_{YY}(\omega) = S_{XX}(\omega) \left| \frac{\sin(T\omega)}{T\omega} \right|^2$$

Note that $h(t)$ acts as a crude low-pass filter that attenuates high-freq. power in the signal.



Given a W.S.S RP $X(t)$ that is the input to an L.T.I system:

$$X(t) \rightarrow \boxed{h(t)} \rightarrow Y(t)$$

If I want to find the autocorrelation function of the output I have two options:

$$(1) R_Y(\tau) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_X(\tau - \alpha + \beta) h(\alpha) h(\beta) d\alpha d\beta$$

Most of the time this is a complicated calculation.

(2) Use $S_{YY}(\omega) = |H(\omega)|^2 \cdot S_{XX}(\omega)$

(a) Compute

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_X(\tau) e^{-i\omega\tau} d\tau$$

(b) Compute

$$H(\omega) = \int_{-\infty}^{\infty} h(t) e^{-i\omega t} dt$$

(c) Compute

$$S_{YY}(\omega) = S_{XX}(\omega) \cdot |H(\omega)|^2$$

(d) Compute

$$R_Y(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{YY}(\omega) e^{+i\omega\tau} d\omega$$

Often the inverse Fourier transform can be looked up in a table:

TABLE 10-1*

$$R(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) e^{i\omega\tau} d\omega \leftrightarrow S(\omega) = \int_{-\infty}^{\infty} R(\tau) e^{-i\omega\tau} d\tau$$

$$\delta(\tau) \leftrightarrow 1$$

$$1 \leftrightarrow 2\pi\delta(\omega)$$

$$e^{i\beta\tau} \leftrightarrow \frac{1}{2\pi}\delta(\omega - \beta)$$

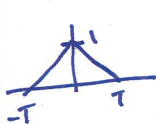
$$\cos \beta\tau \leftrightarrow \pi\delta(\omega - \beta) + \pi\delta(\omega + \beta)$$

$$e^{-\alpha|\tau|} \leftrightarrow \frac{2\alpha}{\alpha^2 + \omega^2}$$

$$e^{-\alpha\tau^2} \leftrightarrow \sqrt{\frac{\pi}{\alpha}} e^{-\omega^2/4\alpha}$$

$$e^{-\alpha|\tau|} \cos \beta\tau \leftrightarrow \frac{\alpha}{\alpha^2 + (\omega - \beta)^2} + \frac{\alpha}{\alpha^2 + (\omega + \beta)^2}$$

$$2e^{-\alpha\tau^2} \cos \beta\tau \leftrightarrow \sqrt{\frac{\pi}{\alpha}} [e^{-(\omega - \beta)^2/4\alpha} + e^{-(\omega + \beta)^2/4\alpha}]$$

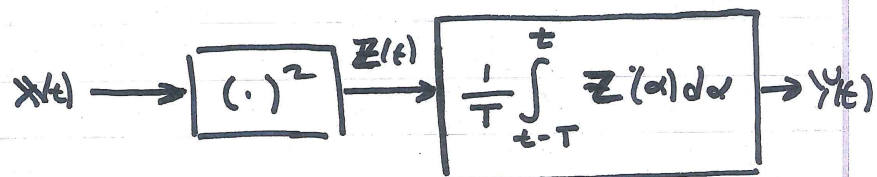


$$\begin{cases} 1 - \frac{|\tau|}{T} & |\tau| < T \\ 0 & |\tau| > T \end{cases} \leftrightarrow \frac{4 \sin^2(\omega T/2)}{T\omega^2}$$

$$\frac{\sin \sigma\tau}{\pi\tau} \leftrightarrow \begin{cases} 1 & |\omega| < \sigma \\ 0 & |\omega| > \sigma \end{cases}$$

* This table will be included in the final exam.

Aside: A common way to estimate the instantaneous power in a signal is to cascade a memoryless system and a L.T.I. System



L.T.I. System
with

$$h(t) = \frac{1}{T} \mathbf{1}_{[0, T]}(t)$$

Memoryless Systems

Defn: A system is called memoryless if its output at time t :

$$y(t) = T[x(t)] = g(x(t)),$$

where $g: \mathbb{R} \rightarrow \mathbb{R}$, is only a function only of the current value of $x(t)$.

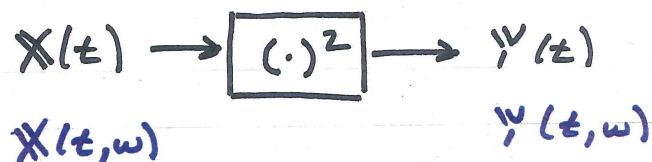
- $g(\cdot)$ is not a function of past or future values of its input
- $y(t) = g(x(t))$ depends only on the instantaneous value of $x(t)$ at time t .

Ex.1. $g(x) = x^2$

30.31

$$Y(t) = g(X(t)) = X^2(t)$$

is a memoryless system.



Ex.2 Integrators are not memoryless. They have memory of the past.

$$Y(t) = \int_{-\infty}^t X(\alpha) d\alpha$$

Block diagram of an integrator system:

$$X(t) \rightarrow \boxed{\int_{-\infty}^t (\cdot) dt} \rightarrow Y(t)$$

$$Y(t, \omega) = \int_{-\infty}^t X(\alpha, \omega) d\alpha$$

For a memoryless system with

30.32

$$Y(t) = g(X(t)),$$

the first order density $f_{Y(t)}(y)$ of $Y(t)$ can be expressed in terms of the first order density $f_{X(t)}(x)$ of $X(t)$ and $g(\cdot)$

$X(t)$ is just a R.V. and

$Y(t) = g(X(t))$ is just a transformed RV (function of $X(t)$.)

Note also that

$$E[Y(t)] = E[g(X(t))] = \int_{-\infty}^{\infty} g(x) f_{X(t)}(x) dx$$

and

$$R_{YY}(t_1, t_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x_1) g(x_2) \cdot f_{X(t_1), X(t_2)}(x_1, x_2) dx_1 dx_2$$

Also, you can get the n -th order pdf of $Y(t)$ using the mapping

$$Y(t_1) = g(X(t_1)), Y(t_2) = g(X(t_2)), \\ \dots, Y(t_n) = g(X(t_n)).$$

Theorem: Let $X(t)$ be a S.S.S. R.P. that is the input to a memoryless system. Then the output is also a S.S.S. R.P.

Proof: Papoulis P.394 (P.304 in 3rd Ed.)

$$f_{Y(t_1+c), \dots, Y(t_n+c)}(y_1, \dots, y_n) = f_{X(t_1+c), \dots, X(t_n+c)}(x_1(y), \dots, x_n(y)) \left| \frac{\partial(x_1, \dots, x_n)}{\partial(y_1, \dots, y_n)} \right|$$

$$= f_{X(t_1), \dots, X(t_n)}(x_1(y), \dots, x_n(y)) \left| \frac{\partial(x_1, \dots, x_n)}{\partial(y_1, \dots, y_n)} \right| = f_{Y(t_1), \dots, Y(t_n)}(y_1, \dots, y_n)$$

not a fun. of c .

$\forall c \in \mathbb{R}, \forall n \in \mathbb{N}$ and all $t_1, \dots, t_n$.

n.b. Regarding the invariance of the Jacobian with a change in time origin:
Assuming $g(\cdot)$ is invertible and $h(\cdot) = g^{-1}(\cdot)$

30.35

$$\begin{aligned} x_1 &= h(y_1) \\ x_2 &= h(y_2) \\ &\vdots \\ x_n &= h(y_n) \end{aligned} \Rightarrow \frac{\partial(x_1, \dots, x_n)}{\partial(y_1, \dots, y_n)} = \begin{vmatrix} \frac{\partial h(y_1)}{\partial y_1} & 0 & 0 & \dots & 0 \\ 0 & \frac{\partial h(y_2)}{\partial y_2} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \frac{\partial h(y_n)}{\partial y_n} \end{vmatrix}$$

$$= \frac{\partial h(y_1)}{\partial y_1} \cdot \frac{\partial h(y_2)}{\partial y_2} \cdot \dots \cdot \frac{\partial h(y_n)}{\partial y_n}$$

Since $g(x)$ does not vary with time, neither does $h(y) = g^{-1}(y)$, so the Jacobian does not depend on time.

Example: Hard Limiter

30.36

Consider a memoryless system with

$$g(x) = \begin{cases} +1 & , x \geq 0, \\ -1 & , x < 0 \end{cases}$$

Consider $y(t) = g(x(t)) = \text{sgn}(x(t))$

Find $E[y(t)]$ and $R_{yy}(t_1, t_2)$ given the "statistics" of $x(t)$.

30.37

$$\begin{aligned}
 E[\psi(t)] &= (+1) \cdot P(\xi \psi(t) = 1) \\
 &\quad + (-1) \cdot P(\xi \psi(t) = -1) \\
 &= (+1) \cdot P(\xi \chi(t) \geq 0) + (-1) \cdot P(\xi \chi(t) < 0) \\
 &= 1 - F_{\chi(t)}(0) - F_{\chi(t)}(0) \\
 &= 1 - 2F_{\chi(t)}(0).
 \end{aligned}$$

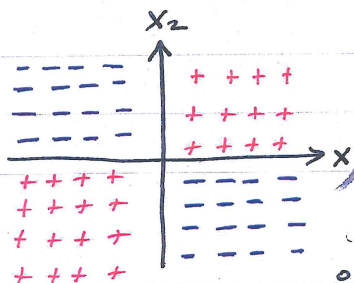
$$R_{\psi\psi}(t_1, t_2) = E[\psi(t_1) \cdot \psi(t_2)]$$

30.38

note: $\psi(t_1) \cdot \psi(t_2) = \begin{cases} +1, & \chi(t_1) \chi(t_2) > 0 \\ -1, & \chi(t_1) \chi(t_2) \leq 0 \end{cases}$

So

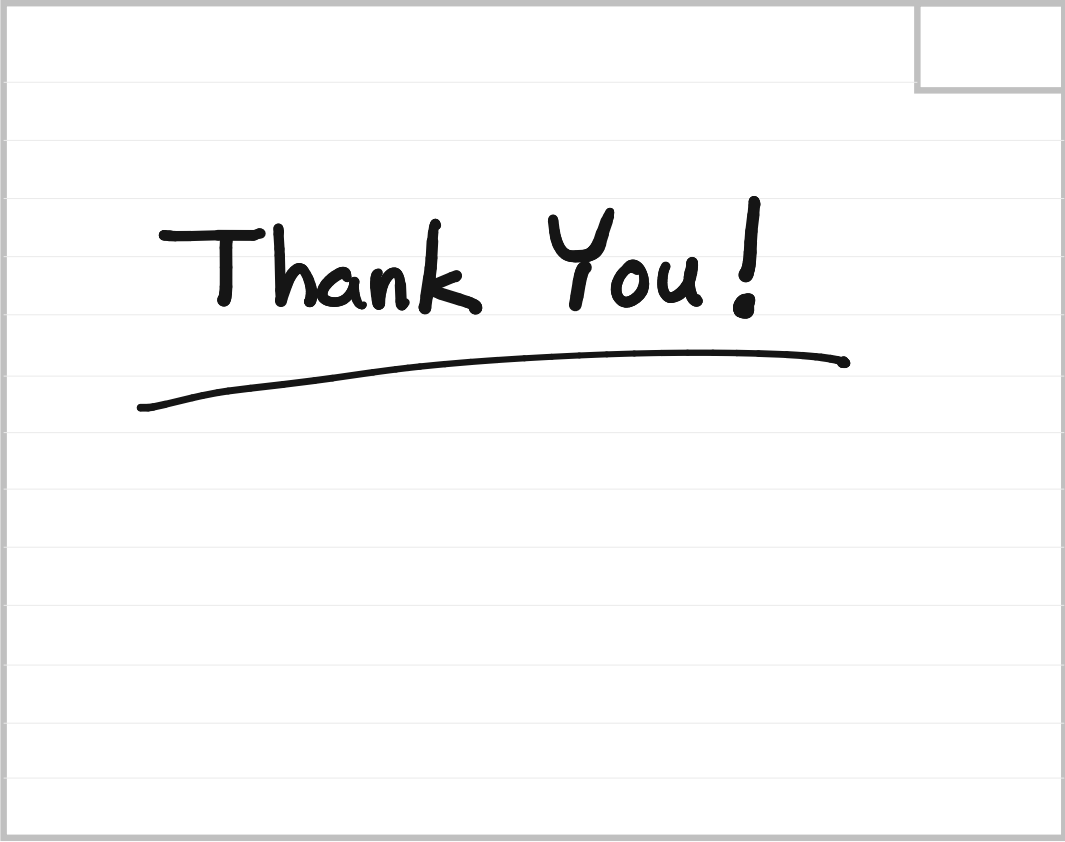
$$\begin{aligned}
 E[\psi(t_1) \psi(t_2)] &= 1 \cdot P(\xi \chi(t_1) \chi(t_2) \geq 0) \\
 &\quad - 1 \cdot P(\xi \chi(t_1) \chi(t_2) < 0)
 \end{aligned}$$



Where

$$\begin{aligned}
 P(\xi \chi(t_1) \chi(t_2) \geq 0) &= \int_0^\infty \int_0^\infty f_{\chi(t_1), \chi(t_2)}(x_1, x_2) dx_1 dx_2 \\
 &\quad + \int_{-\infty}^0 \int_{-\infty}^0 f_{\chi(t_1), \chi(t_2)}(x_1, x_2) dx_1 dx_2
 \end{aligned}$$

and $P(\xi \chi(t_1) \chi(t_2) < 0) = \int_0^\infty \int_{-\infty}^0 f_{\chi(t_1), \chi(t_2)}(x_1, x_2) dx_1 dx_2 + \int_{-\infty}^0 \int_0^\infty f_{\chi(t_1), \chi(t_2)}(x_1, x_2) dx_1 dx_2$



Thank You!