Session 30

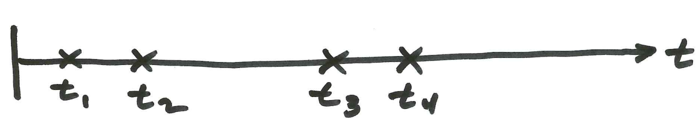
The Poisson Point Process

30.1

Defn: A point process is a set of random points $\{t_i\}$ on the time axis.

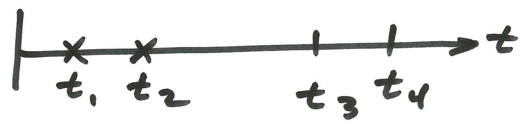
n.b. The "points" are RVs that represent times at which some random "events" in time occur.

(e.g. times at which light bulbs in your house burn out.)



30.2

A point process is a set of random points along the time axis: $\{t_i\}$



$$\underbrace{t_1 \le t_2 \le t_3 \le \cdots \le t_n \le \cdots}$$

The points are ordered in time.

The $\{t_i\}$ are a collection of RVs defined on some probability space $(\mathcal{S}, \mathcal{F}, P)$.

30.3

<u>Defn</u>: To each point process $\{t_i\}$ we can assign a random process $X(t)$ called the <u>counting process</u>, defined as

$$X(t) \triangleq \text{number of points in the interval } [0, t).$$

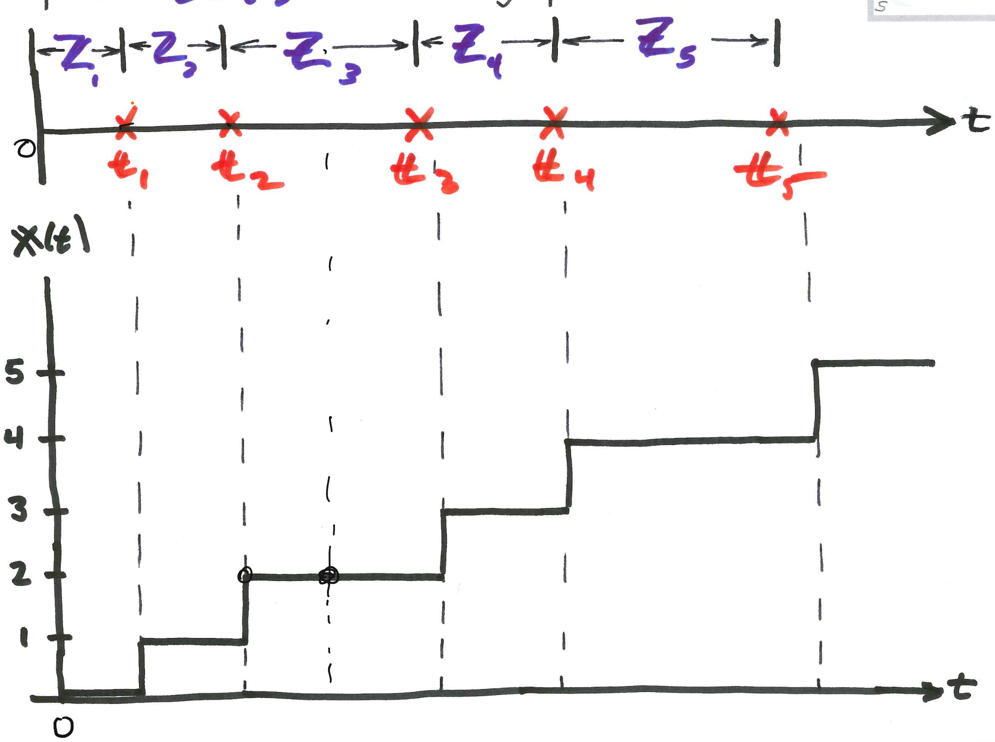
Defn: To any point process $\{t_i\}$ we can associate a <u>renewal process</u>, a sequence of RVs defined by

$$Z_n = \begin{cases} t_1 & , n=1 \\ t_n - t_{n-1} & , n=2,3,\ldots \end{cases}$$

The renewal process represents the time between the events in the point process $\{t_i\}$.

The $\{Z_n\}$ are RVs defined on $(\mathcal{S}, \mathcal{F}, P)$.

Relationship between point process $\{t_i\}$, renewal process $\{Z_i\}$ and counting process $X(t)$:



Defn: A point process $\{t_i\}$ is called a Poisson point process or set of Poisson points if

1. The number of points $N(t_1, t_2)$ in the time interval $[t_1, t_2)$ is a Poisson RV with mean $\lambda(t_2 - t_1)$, where $\lambda > 0$:
$$P(\{N(t_1,t_2)=k\}) = \frac{e^{-\lambda(t_2-t_1)}[\lambda(t_2-t_1)]^k}{k!}$$
for $k = 0, 1, 2, \ldots$

2. If $[t_1, t_2) \cap [t_3, t_4) = \phi$, then $N(t_1, t_2)$ and $N(t_3, t_4)$ are statistically independent RVs. (for real numbers t_1, t_2, t_3, t_4)

n.b. The mean is proportional to the length of the interval (i.e., the mean of $N(t_1, t_2)$.)

* Technically, this is a homogeneous Poisson process with constant rate λ.

(This can be generalized to a nonhomogeneous Poisson process where the rate $\lambda(t)$ is not constant. Here $N(t_1, t_2)$ has mean $\Lambda(t_1, t_2) = \int_{t_1}^{t_2} \lambda(t)\,dt$ we will not consider this)

30.8

Defn: The random process

$$X(t) \triangleq N(0,t)$$

corresponding to a set of Poisson points $\{t_i\}$ is called a **Poisson Counting process**

See Ex. 9-5 (10-5 in 3rd. Edition) of Papoulis.

For this R.P. $X(t)$, it can be shown that

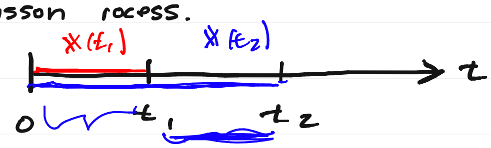
$$E[X(t)] = \lambda t$$

$$R_{XX}(t_1, t_2) = \lambda \cdot \min(t_1, t_2) + \lambda^2 t_1 t_2.$$

30.9

n.b. In computing $R_{XX}(t_1,t_2) = E[X(t_1)X(t_2)]$ we will use the independent increment property (prop. 2) of the Poisson rocess.

Assume $t_2 > t_1$:



$$E[X(t_1)X(t_2)] = E[N(0,t_1) \cdot N(0,t_2)]$$

$$= E[X(t_1)(X(t_1) + (X(t_2) - X(t_1)))]$$

$$= E[X^2(t_1)] + E[X(t_1)(X(t_2) - X(t_1))]$$

$$= E[X^2(t_1)] + E[N(0,t_1) \cdot N(t_1,t_2)]$$

$$= E[X^2(t_1)] + E[N(0,t_1)] \cdot E[N(t_1,t_2)]$$

$$= \ldots = \lambda t_1 + \lambda^2 t_1 t_2$$

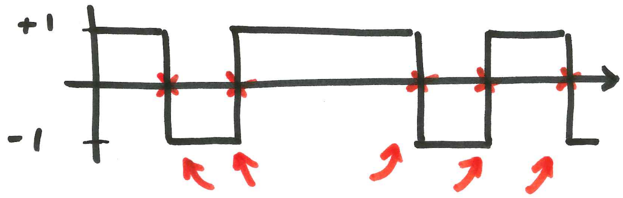
By symmetry, if $t_1 > t_2$

$$E[X(t_1)X(t_2)] = \lambda t_2 + \lambda^2 t_1 t_2.$$

30.10

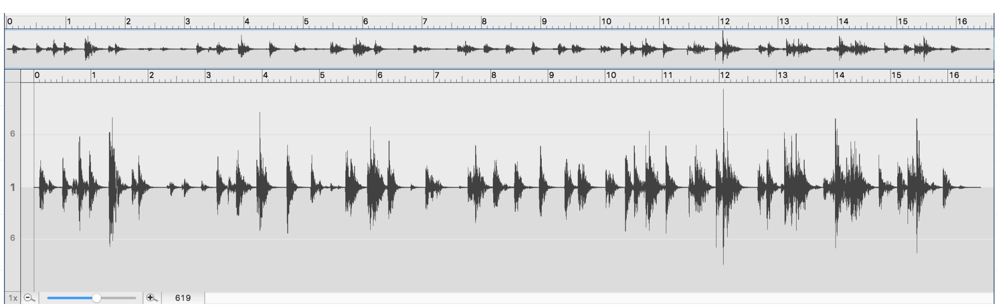
n.b. The Poisson process is used to generate the Random Telegraph Process

e.g.:



Poisson Points determine transitions

Microwave Popcorn Recording



30.11

Q: What is the pdf of the n-th point t_n of a Poisson process?

CDF: $F_{t_n}(t) = P(\{t_n \le t\})$, $t > 0$.

Note that

$$\{t_n \le t\} \Longleftrightarrow \{N(0,t) \ge n\}$$

$$\Rightarrow \quad \{t_n \le t\} = \{N(0,t) \ge n\}$$

30.12

$$F_{t_n}(t) \triangleq P(\{t_n \le t\}) = P(\{N(0,t) \ge n\})$$
$$= 1 - P(\{N(0,t) < n\})$$
$$= 1 - P(\{N(0,t) \le n-1\})$$
$$= 1 - \sum_{k=0}^{n-1} \frac{e^{-\lambda t}(\lambda t)^k}{k!}$$

Thus, the pdf of t_n is

$$f_{t_n}(t) = \frac{d}{dt}\left(F_{t_n}(t)\right)$$
$$= \frac{d}{dt}\left(1 - \sum_{k=0}^{n-1} \frac{e^{-\lambda t}(\lambda t)^k}{k!}\right)$$
$$= \frac{d}{dt}\left(\sum_{k=n}^{\infty} \frac{e^{-\lambda t}(\lambda t)^k}{k!}\right)$$

30.13

$$\cdots \sum_{k=n}^{\infty} \underbrace{\frac{d}{dt}\left(\frac{e^{-\lambda t}(\lambda t)^k}{k!}\right)}_{\Downarrow} \quad --- \quad (*)$$

Now $\dfrac{d}{dt}\left(\dfrac{e^{-\lambda t}(\lambda t)^k}{k!}\right) = -\dfrac{\lambda e^{-\lambda t}(\lambda t)^k}{k!} + \dfrac{k e^{-\lambda t}(\lambda t)^{k-1}\lambda}{k!}$

$$= \lambda e^{-\lambda t}\left[\frac{(\lambda t)^{k-1}}{(k-1)!} - \frac{(\lambda t)^k}{k!}\right] \longleftarrow$$

So, substituting this expression back into $(*)$, we get

30.14

$$f_{t_n}(t) = \lambda e^{-\lambda t} \sum_{k=n}^{\infty}\left(\frac{(\lambda t)^{k-1}}{(k-1)!} - \frac{(\lambda t)^k}{k!}\right)$$

$$= \lambda e^{-\lambda t}\left(\underbrace{\frac{(\lambda t)^{n-1}}{(n-1)!} - \frac{(\lambda t)^n}{n!}}_{k=n} + \underbrace{\frac{(\lambda t)^n}{n!} - \frac{(\lambda t)^{n+1}}{(n+1)!}}_{k=n+1} + \underbrace{\frac{(\lambda t)^{n+1}}{(n+1)!} - \frac{(\lambda t)^{n+1}}{(n+2)!}}_{k=n+2} + \cdots\right)$$

0 0 0

$$\Downarrow$$

$$\lambda e^{-\lambda t}\left(\frac{(\lambda t)^{n-1}}{(n-1)!}\right)$$

$$\therefore \quad f_{t_n}(t) = \frac{\lambda(\lambda t)^{n-1} e^{-\lambda t}}{(n-1)!}, \quad t \geq 0.$$

$\therefore$ The n-th point t_n of a homogeneous Poisson process with rate λ is

$$\boxed{f_{t_n}(t) = \frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!} \cdot 1_{[0,\infty)}(t)}$$

This is the $\underline{n\text{-th order Erlang pdf}}$

(Agner Kraup Erlang – Danish mathematician and engineer)

It is a special case of the gamma pdf:

$$f_X(x) = \frac{\beta^\alpha x^{\alpha-1} e^{-\beta x}}{\Gamma(\alpha)} 1_{[0,\infty)}(x) \quad \left(\phi_X(s) = \frac{\beta^\alpha}{(s+\beta)^\alpha} \right)$$

when $\alpha = n$ and $\beta = \lambda$.

(n.b. $\Gamma(n) = (n-1)!$ for $n \in \mathbb{N}$)

For $n=1$, this becomes

$$f_{t_1}(t) = \lambda e^{-\lambda t} \cdot 1_{[0,\infty)}(t)$$

$\Rightarrow$ t_1 is exponentially distributed with mean $1/\lambda$.

$\Rightarrow$ $Z_n = t_n - t_{n-1}$ is exponentially distributed with mean $1/\lambda$ when $n = 2, 3, \ldots$

(i.e., the time between Poisson Points is exponentially distributed with mean $1/\lambda$.)

Simulating a Homogeneous Poisson Process

We can use the fact that the time between Poisson points in a homogeneous Poisson process with rate λ are independent identically distributed exponentially distributed RVs with mean $1/\lambda$ to simulate a homogeneous Poisson Process:

1. Generate i.i.d exponential RVs with mean $\mu = 1/\lambda$: $Z_1, Z_2, \ldots, Z_n$
2. Generate the Poisson points

$$t_k = \sum_{m=1}^{k} Z_m \quad , k = 1, \ldots, n$$

2. Generate the Poisson points

$$t_k = \sum_{m=1}^{k} Z_m \quad , k = 1, \ldots, n$$

Then we have that $\{t_1, t_2, \ldots, t_n\}$ are the first n Poisson points of a Poisson point process.

The i.i.d. exponential RVs with mean $1/\lambda$

$$Z_1, Z_2, \ldots, Z_n, \ldots$$

are the renewal process corresponding to the Poisson points $t_1, t_2, \ldots, t_n, \ldots$

The End!

* Please fill out the course evaluations for the course.

n.b. Overflow Section will take the final exam with FNY Section in G140

30.19

If you are looking for additional problems to solve for practice for the final exam, here is one source:

Hwei P. Hsu, *Probability, Random Variables, and Random Processes*, 3rd Edition, Schaum's Outline, McGraw-Hill, 2014

An electronic (Kindle) version is available for instant download on Amazon ($14.16)

406 Fully solved problems.

ECE 600 Final Exam

Monday, April 29, 2024
3:30 to 5:30 pm
Room FNY G140 { For FNY and ONC (overflow) Sections

This will be a comprehensive exam.

This is a closed book, closed notes exam.

You may <u>not</u> use a calculator.

Start each problem on a new page.

- 10 "simple" multiple choice* questions **50%**
  * There will be room to show your work.
- Two "work-out" problems **34%**
- One page of True/False Questions **16%**

EPE Online Section Answer Form

ECE600 Random Variables and Waveforms
Spring ~~2022~~ 2024

Mark R. Bell
MSEE 336

Final Exam Online Section Answer Form

~~May 4, 2022~~
April 29, 2024

1–10	
11	
12	
13	
Total	

$$R(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)e^{j\omega\tau}\,d\omega \leftrightarrow S(\omega) = \int_{-\infty}^{\infty} R(\tau)e^{-j\omega\tau}\,d\tau$$

$$\delta(\tau) \leftrightarrow 1 \qquad 1 \leftrightarrow 2\pi\delta(\omega)$$

$$e^{j\beta\tau} \leftrightarrow 2\pi\delta(\omega-\beta) \qquad \cos\beta\tau \leftrightarrow \pi\delta(\omega-\beta)+\pi\delta(\omega+\beta)$$

$$e^{-\alpha|\tau|} \leftrightarrow \frac{2\alpha}{\alpha^2+\omega^2} \qquad e^{-\alpha\tau^2} \leftrightarrow \sqrt{\frac{\pi}{\alpha}}\,e^{-\omega^2/4\alpha}$$

$$e^{-\alpha|\tau|}\cos\beta\tau \leftrightarrow \frac{\alpha}{\alpha^2+(\omega-\beta)^2}+\frac{\alpha}{\alpha^2+(\omega+\beta)^2}$$

$$2e^{-\alpha\tau^2}\cos\beta\tau \leftrightarrow \sqrt{\frac{\pi}{\alpha}}\left[e^{-(\omega-\beta)^2/4\alpha}+e^{-(\omega+\beta)^2/4\alpha}\right]$$

$$\begin{cases} 1-\frac{|\tau|}{T} & |\tau|<T \\ 0 & |\tau|>T \end{cases} \leftrightarrow \frac{4\sin^2(\omega T/2)}{T\omega^2}$$

$$\frac{\sin\sigma\tau}{\pi\tau} \leftrightarrow \begin{cases} 1 & |\omega|<\sigma \\ 0 & |\omega|>\sigma \end{cases}$$

Name: ____________________

ID #: ____________________

Directions:

1. **Print** your name and student number on the cover page.
2. Exam is closed book, closed notes, and no calculators.
3. Clearly designate all answers asked for (arrows, underline, box, etc.)

Problems 1–10 are multiple choice problems worth 5 points each. For each problem, write the lettercorresponding to the best answer next to the problem number. Space is provided to work out your solution for each of these problems. Please show your work! If your final grade is near a borderline, the quality of your written solutions could significantly impact your final grade.

1. Answer: $\boxed{C}$

$$\Phi_X(\omega) = E\left[e^{i\omega X}\right]$$

$$= \sum_{k=0}^{n} e^{i\omega k}\binom{n}{k}p^k(1-p)^{n-k}$$

$$= \sum_{k=0}^{n}\binom{n}{k}(pe^{i\omega})^k(1-p)^{n-k}$$

Binomial Thm.

$$= (pe^{i\omega}+1-p)^n$$

$$= (1+p(e^{i\omega}-1))^n$$

etc.

2. Answer:

11. Problems 11 is made up of 8 True/False questions, worth 2 points each. Fill in your answers T (true) or F (false) below, corresponding to the statements A–H in problem 11 on the exam.

A. ____

B. ____

C. ____

D. ____

E. ____

F. ____

G. ____

H. ____

Problems 12 and 13 are “work out” problems for which partial credit will be awarded for correctly reasoned work. It is important that you coherently present your thinking in the solution of these problems if you wish to receive partial credit (or full credit for that matter.) Please work problems 12 and 13 of the exam in the designated space below.

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12. Problem 12 Solution:

ECE 600, Final Exam *Name:* ____________

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13. Problem 13 Solution:

Thank You!