Session 3

Recall...

The Sample Space $\underline{S}$

<u>Defn</u>: The sample space $\underline{S}$ is a non-empty set of possible outcomes of a random experiment.

One and only one outcome from the sample space occurs when we perform a random experiment.

The Event Space $\mathcal{F}(\mathcal{S})$

Defn: The <u>event space</u> $\mathcal{F}(\mathcal{S})$ is a non-empty collection of subsets of $\mathcal{S}$ satisfying the following closure properties.

1. If $- \in \mathcal{F}(\mathcal{S})$, then $\bar{A} \in \mathcal{F}(\mathcal{S})$.

2. For any finite n, if $A_i \in \mathcal{F}(\mathcal{S})$ for $i = 1, 2, \ldots, n$, then

$$\bigcup_{i=1}^{n} A_i \in \mathcal{F}.$$

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3. If $A_i \in \mathcal{F}(\mathcal{S})$, $i = 1, 2, 3, \ldots$

then

$$\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}(\mathcal{S}).$$

A sets of subsets satisfying these 3 properties is called a <u>σ-field</u>.

(n.b. If only 1 and 2 hold, you have a <u>field</u> of sets.)

Probability Measure

Defn: A probability measure $P(\cdot)$ (corresponding to $\mathcal{S}$ and $\mathcal{F}(\mathcal{S})$) is an assignment of a real number $P(A)$ to each $A \in \mathcal{F}(\mathcal{S})$ satisfying the Axioms of Probability

Axioms of Probability

1. $P(A) \geq 0$, $\forall A \in \mathcal{F}(\mathcal{S})$.

2. $P(\mathcal{S}) = 1$.

3. If $A_1, A \in \mathcal{F}(\mathcal{S})$ and $A_1 \cap A_2 = \phi$,
   then $P(A_1 \cup A_2) = P(A_1) + P(A_2)$

- If $\{A_1, \ldots, A_n\}$ (finite) are $\underline{\text{disjoint}}$ $(A_j \cap A_k = \phi)_{j \neq k}$
$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i)$$

4. If $A_1, A_2, \ldots, A_n, \ldots \in \mathcal{F}(\mathscr{S})$ is a countable collection of disjoint events, then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i).$$

n.b. $P(\cdot)$ is a set function.

$P(\cdot): \mathcal{F}(\mathscr{S}) \rightarrow \mathbb{R}.$

If we want to talk about the probability of a particular outcome $w_0 \in \mathscr{S}$, we do so by considering the singleton set $\{w_0\} \in \mathcal{F}(\mathscr{S})$

$P(\{w_0\})$ is well defined

$P(w_0)$ is **not** well defined

Examples of $(S, \mathcal{F}, P)$:

Ex. 1 $S = \{0, 1\}$

$\mathcal{F} = \{\phi, S, \{0\}, \{1\}\}$

$$P(A) = \begin{cases} \alpha, & A = \{0\} \\ 1-\alpha, & A = \{1\} \\ 0, & A = \phi \\ 1, & A = S \end{cases}$$

where $0 \le \alpha \le 1$.

P(A) satisfies the axioms of probability.

Ex. 2: Let S be any sample space.

Let $\mathcal{F}(S) = \{\phi, S\}$

By the axioms of prob., there is only one prob. measure $P(\cdot)$ we can assign:

$$P(A) = \begin{cases} 1, & A = S \\ 0, & A = \phi \end{cases}$$

Empty sets: Good notation: ϕ and $\{\}$

Bad notation: $\{\emptyset\}$

It follows from the axioms of probability that

$$P(\phi) = 0.$$

Proof: $\qquad S = S \cup \phi$

$\qquad\qquad\qquad \underset{\text{disjoint}}{\uparrow\quad\uparrow} \quad \sim S \cap \phi = \phi$

$\Rightarrow \quad P(S) = P(S \cup \phi) = \underbrace{P(S)} + P(\phi)$

$\qquad\qquad \downarrow$

$\qquad 1 \quad = \qquad\qquad\qquad 1 \quad + P(\phi)$

$\Rightarrow \quad P(\phi) = 1 - 1 = 0 .$

Similarly, it follows from the axioms of probability that

$$P(\bar{A}) = 1 - P(A).$$

Proof: $\qquad S = A \cup \bar{A}$

$\qquad\qquad\qquad \underset{\text{disjoint}}{\uparrow\quad\uparrow}$

$1 \overset{ax.2}{=} P(S) = P(A \cup \bar{A}) \overset{ax.3}{=} P(A) + P(\bar{A})$

$\Rightarrow \quad P(A) + P(\bar{A}) = 1$

$\Rightarrow \quad P(\bar{A}) = 1 - P(A) .$

Brief Summary:

- Random Experiment $\Rightarrow$ random outcome
- Ω is the set of all possible outcomes.
- Events are described as subsets of Ω.
- If $A \subset \Omega$, we say that event A has occured if the random outcome $\omega \in \Omega$ is in A.
- Events of interest are collected into the event space $\mathcal{F}(\Omega)$. $\mathcal{F}(\Omega)$ also includes subsets generated by the closure properties. (σ-field).

- The probability that an event $A \in \mathcal{F}(\Omega)$ occurs is given by $P(A)$

$$P(\cdot) : \mathcal{F}(\Omega) \longrightarrow \mathbb{R}$$

and satisfies the axioms of probability

We now take a more detailed look at $\mathcal{S}$, $\mathcal{F}$, and P:

Sample Space, $\mathcal{S}$:

Intuitively : Set containing all possible outcomes of a random experiment.

Mathematically : Universal set (set of all outcomes.)

Ex.1 : A finite sample space:

$$\mathcal{S} = \{ w_k : k = 1, 2, \ldots, n \}$$

specifically : $\mathcal{S} = \{0, 1\}$, $\mathcal{S} = \{H, T\}$
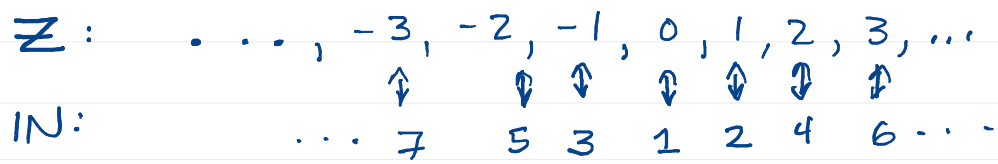
$$\mathcal{S} = \{1, 2, 3, 4, 5, 6\}$$

Ex.2 : A countable sample space:

$$\mathcal{S} = \{ w_k : k = 1, 2, 3, \ldots \}$$

specifically : $\mathbb{N} = \{1, 2, 3, \ldots \}$

$$\mathbb{Z}_+ = \{0, 1, 2, \ldots \}$$

$$\mathbb{Z} = \{ \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots \}$$

$\mathbb{Z}$ is countable because it can be put into one-to-one correspondence with IN:

$\mathbb{Z}$:   . . . , -3, -2, -1, 0, 1, 2, 3, . . .

IN: . . . 7 5 3 1 2 4 6 . . .

The Rational Numbers are Countable

Consider $\frac{31}{255}$. It sits in the array at the intersection of row 31 and column 255.

EX.3 $\mathscr{S} = (\alpha, \beta)$, $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$.

$\qquad = \{x \in \mathbb{R} : \quad \alpha < x < \beta\}$

This is an uncountable set.

$\qquad [\alpha, \beta] = \{x \in \mathbb{R} : \alpha \leq x \leq \beta\}$
$\qquad \qquad$ is uncountabb.

$\qquad \mathbb{R} = (-\infty, +\infty)$ is uncountable.

$*$ Let $\alpha = 0$ and $\beta = 1$
$\qquad \Rightarrow [0,1]$ is uncountable.

Ex.4   Let $\mathscr{S}$ be the set of all k-dim vectors whose coordinate (elements) come from any of the examples Ex.1 – Ex.3

(Call the "$\mathscr{S}$" of the previous example A)

$\qquad \mathscr{S} = \underbrace{A \times A \times \cdots \times A}_{\substack{k\text{-fold} \\ \text{Cartesian product}}} = \prod_{i=1}^{k} A$

Examples: – State of a control system ($A = \mathbb{R}$).
$\qquad \qquad$ – Length k binary codeword ($A = \{0, 1\}$)
$\qquad \qquad$ – Length k discrete-time signal
$\qquad \qquad \qquad$ ($\mathscr{S} = \mathbb{R}$ or $\mathbb{C}$).

How can we see that the set $[\alpha, \beta]$ is uncountable?

Lets look at $[0,1]$.

Prove this is uncountable.

Assume $[0,1]$ is countable. Then we can list the elements of $[0,1]$ in one-to-one correspondence with the natural numbers.

$$\frac{N}{①} \longleftrightarrow 0.d^{(1)} \quad d_2^{(1)} \quad d_3^{(1)} \cdots$$

$$2 \longleftrightarrow 0.d_1^{(2)} \quad d_2^{(2)} \quad d_3^{(2)} \cdots$$

$$3 \longleftrightarrow 0.d_1^{(3)} \quad d_2^{(3)} \quad d_3^{(3)} \cdots$$

$$\vdots$$

$d_j^{(k)} = \{0,1\}$ the value of the k-th number is

$$\sum_{j=1}^{\infty} \frac{d_j^{(k)}}{2^j}$$

Consider the diagonal $d_j^{(j)}$ for $j = 1, 2, 3, \ldots$

What about the number

$$0.\tilde{d}_1 \, \tilde{d}_2 \, \tilde{d}_3 \cdots$$

where $\tilde{d}_k \neq d_k^{(k)}$

$\tilde{d}_k \in \{0,1\}$

Cantor Diagonalization Method

So the number

$$0.\tilde{d}_1 \, \tilde{d}_2 \, \tilde{d}_3 \cdots = \sum_{j=1}^{\infty} \frac{\tilde{d}_j}{2^j} \in [0,1]$$

But it's not in the original countable list

$\Rightarrow\Leftarrow$ Contradiction $\Rightarrow [0,1]$ is not countable

i.e., $[0,1]$ is uncountable.

Ex.5 A space of countable sequences drawn from Ex.1 – Ex.3

$$\mathcal{S} = A \times A \times \cdots \times A \times \cdots$$

$$= \prod_{i \in \mathbb{N}} A = \prod_{i=1}^{\infty} A = A^{\mathbb{N}}$$