Session 29

Recall...

Stochastic Processes

Defn(1): A random process defined on a random experiment $(\mathcal{S}, \mathcal{F}, P)$ is a family $\{X(t);\ t \in \mathbb{R}\}$ of random variables defined on $(\mathcal{S}, \mathcal{F}, P)$ and indexed by t.

Recall: A RV $X : \mathcal{S} \to \mathbb{R}$ for a particular outcome $w_0 \in \mathcal{S}$ is just a real number $X(w_0)$.

A random process $X(t, w)$ is a function of two variables: time and the outcome $w \in \mathcal{S}$.

Recall...

29.2

Defn : For a fixed outcome $\omega_0 \in \mathcal{S}$, the time function $x(\cdot) = \mathbb{X}(\cdot, \omega_0)$ is called the <u>sample path</u> of the random process $\mathbb{X}(t)$ corresponding to ω_0. Each $\omega \in \mathcal{S}$ has a sample path $x(\cdot) = \mathbb{X}(\cdot, \omega)$. The set of all sample paths

$$\mathcal{E} = \{x(\cdot) : x(\cdot) = \mathbb{X}(\cdot, \omega), \text{ for } \omega \in \mathcal{S}\}$$

is called the <u>ensemble</u> of the R.P. $\mathbb{X}(t)$.

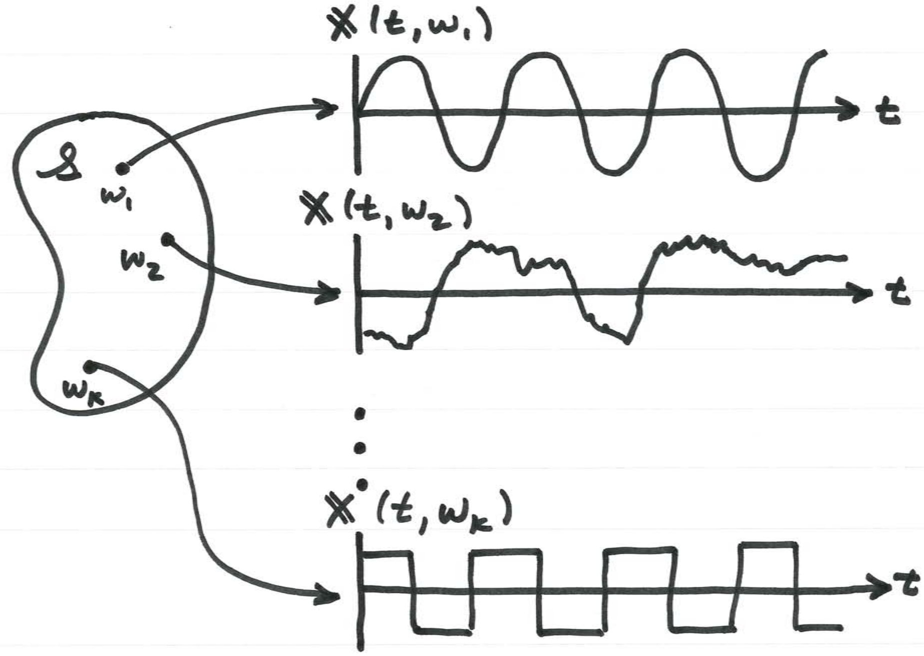
Having established these ideas, we can give the following equivalent definition of a random process ...

Recall...

29.3

Defn$^{(2)}$: Consider a random experiment $(\mathcal{S}, \mathcal{F}, P)$. A function $\mathbb{X} : \mathcal{S} \to \mathcal{E}$, where $\mathcal{E}$ is a set of functions of time (sample paths) is called a <u>random process</u>. That is, $\mathbb{X}$ is a mapping assigning a function of time $\mathbb{X}(\cdot, \omega)$ to each $\omega \in \mathcal{S}$.

A picture appears as follows:



Note that once we know the particular $\omega_k \in \mathcal{S}$ that is the outcome of the experiment, there is nothing random about the random process.

Randomness only enters into the random process through the selection of $\omega \in \mathcal{S}$ in the random experiment.

<u>Note</u>: A random process is completely analogous to a Random variable:

<u>Random Variable</u> $X: \mathcal{S} \rightarrow \mathbb{R}$ (real line)

<u>Random Process</u>: $X(t): \mathcal{S} \rightarrow \mathcal{E}$ (set of functions of time)

29.6

Example: I flip a coin three times (once a second) and construct a random process by making a waveform take on the value during that second corresponding to the outcome of the coin during that second:

$$H \rightarrow -1 \qquad (*)$$
$$T \rightarrow +1$$

I can consider this to be a combined experiment with $2^3 = 8$ possible outcomes:

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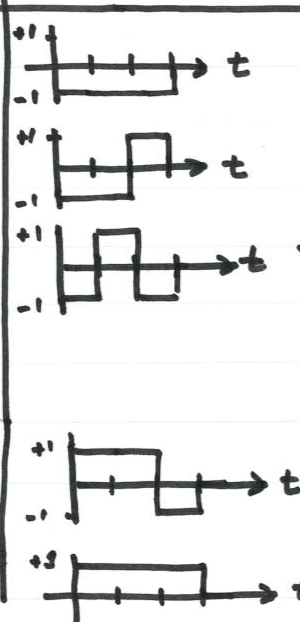
29.7

$\mathcal{S} = \{\omega_1, \dots, \omega_8\}$

ω	
$\omega_1 = (H, H, H)$	$X(t, \omega_1)$
$\omega_2 = (H, H, T)$	$X(t, \omega_2)$
$\omega_3 = (H, T, H)$	$X(t, \omega_3)$
$\vdots$	
$\omega_7 = (T, T, H)$	$X(t, \omega_7)$
$\omega_8 = (T, T, T)$	$X(t, \omega_8)$



Randomness enters in only in the determination of ω by flipping the coin 3 times.

Once you know ω, $X(t, \omega)$ is specified by relation $(*)$ non-randomly.

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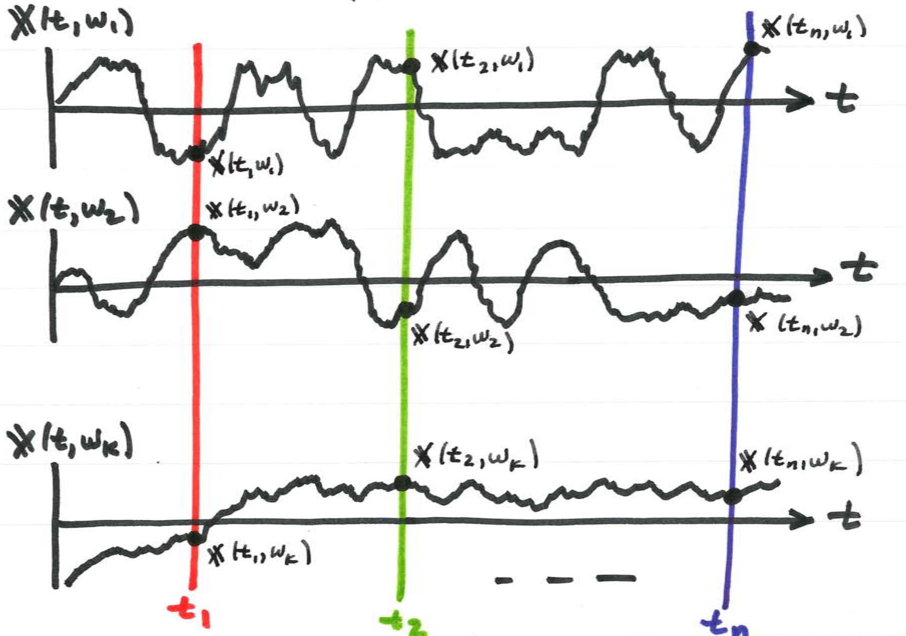
29.8

Note: If we evaluate a random process at time t_1, we get a random variable $X(t_1)$:



If we sample the random process $X(t)$ at n points in time, we get n j-dist RVs $X(t_1), X(t_2), \ldots, X(t_n)$.

29.9

Defn: Fix n time instants $t_1, \ldots, t_n$. Then the <u>n-th order cdf</u> of $X(t)$ at times $t_1, t_2, \ldots, t_n$ is the joint cdf of the n j-dist RVs $X(t_1), X(t_2), \ldots, X(t_n)$:

$$F_{X(t_1)\cdots X(t_n)}(x_1, \ldots, x_n) = P\left(\{X(t_1) \le x_1\} \cap \cdots \cap \{X(t_n) \le x_n\}\right)$$

$$= \text{“} F_{X(t)}(x_1, \ldots, x_n; t_1, \ldots, t_n) \text{”}$$

and the corresponding <u>n-th order pdf</u> of $X(t)$ is

$$f_{X(t_1)\cdots X(t_n)}(x_1, \ldots, x_n) = \frac{\partial^n F_{X(t_1)\cdots X(t_n)}(x_1, \ldots, x_n)}{\partial x_1 \partial x_2 \cdots \partial x_n}$$

$$= \text{“} f_{X(t)}(x_1, \ldots, x_n; t_1, \ldots, t_n) \text{”}$$

Theorem: The probabilistic behavior of a random process $X(t)$ is completely characterized by the collection of all n-th order cdfs or pdfs, for all $t_1, t_2, \ldots, t_n$, and all $n = 1, 2, 3, \ldots$ (all $n \in \mathbb{N}$.)

29.10

Proof: (Kolmogorov) Way beyond this course

But we can use the fact.

Stationary Random Processes

29.11

"Stationary" $\triangleq$ staying in one place throughout time.

What if the n-th order distributions don't change with a shift in the time origin?

" Probabilistic behavior is the same yesterday, today and tomorrow."

Probabilistic description "stays in one place" throughout time.

29.12

Defn: A random process $X(t)$ is called <u>n-th order stationary</u> if all n-th order cdfs or pdfs are invariant to time shifts in the origin:

$$f_{X(t_1)\cdots X(t_n)}(x_1,\ldots,x_n) = f_{X(t_1+c)\cdots X(t_n+c)}(x_1,\ldots,x_n)$$

for all $c \in \mathbb{R}$ and for any n time instants $t_1,\ldots,t_n$.

Defn: A random process $X(t)$ is called <u>stationary</u>, or <u>strict sense stationary</u> (SSS), if $X(t)$ is n-th order stationary for all $n = 1, 2, 3, \ldots$

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29.13

Note: If $X(t)$ is stationary, it is first-order $(n=1)$ stationary.

$$\Rightarrow \quad f_{X(t_1)}(x) = f_{X(t_1+c)}(x), \quad \forall c \in \mathbb{R}$$

$$= f_{X(t_2)}(x), \quad \text{for any } t_2 \in \mathbb{R}$$

$$= f(x \quad \text{(not a function of time)}$$

$$\therefore \quad E[X(t)] = \int_{-\infty}^{\infty} x\, f_{X(t)}(x)\, dx$$

$$= \int_{-\infty}^{\infty} x\, f(x)\, dx = \text{constant}$$

(Not a function of time.)

Similarly

$$\text{var}(X(t)) = E[X^2(t)] - (E[X(t)])^2$$

$$= \sigma^2 = \text{constant}$$

(Not a function of time.)

Example: non-stationary random process.

Let $\mathbb{X}(t) \triangleq e^{-\mathbb{A}t} \cdot 1_{[0,\infty)}(t)$

where $\mathbb{A}$ is a uniformly distributed RV on $[0,1]$.

$\mathbb{X}(t)$ is a random process.
At $t = t_1$, we have that $\mathbb{X}(t_1)$ is a RV:

$$\mathbb{X}(t_1) = e^{-\mathbb{A}t_1} \cdot 1_{[0,\infty)}(t_1) = g(\mathbb{A}, t_1)$$
$$= g_{t_1}(\mathbb{A}).$$

So

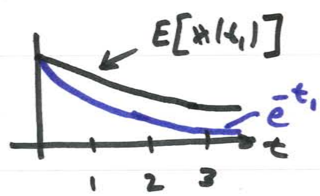
$$E[\mathbb{X}(t_1)] = E[g_{t_1}(\mathbb{A})]$$

$$= \int_{-\infty}^{\infty} g_{t_1}(a) f_{\mathbb{A}}(a)\, da$$

$$= \int_0^1 g_{t_1}(a) \cdot 1\, da = \int_0^1 e^{-at_1}\, da \cdot 1_{[0,\infty)}(t_1)$$

$$= 1_{[0,\infty)}(t_1) \left.\frac{e^{-at_1}}{-t_1}\right|_{a=0}^{1} = \frac{1-e^{-t_1}}{t_1} \cdot 1_{[0,\infty)}(t_1)$$

$$\therefore\ E[\mathbb{X}(t_1)] = \frac{1-e^{-t_1}}{t_1} \cdot 1_{[0,\infty)}(t_1)$$



29.16

Because $E[X(t)]$ is a function of t, it follows that $f_{X(t)}(x)$ must also be a function of t

(you can find $f_{X(t)}(x)$ as an exercise)

$\Rightarrow$ $X(t)$ is <u>not</u> first order stationary

$\Rightarrow$ $X(t)$ is <u>not</u> stationary

29.17

In the special case of <u>2nd order</u> ($n=2$) stationary processes, we have

$$f_{X(t_1)X(t_2)}(x_1, x_2) = f_{X(t_1+c)X(t_2+c)}(x_1, x_2), \quad \forall c \in \mathbb{R}.$$

This means that the second-order pdf

$$f_{X(t_1)X(t_2)}(x_1, x_2) = \text{“} f_X(x_1, x_2; t_1, t_2) \text{”}$$

can depend on t_1 and t_2 only through the time difference $t_1 - t_2$.

In such cases, the correlations between $X(t_1)$ and $X(t_2)$ can depend only on the time difference $t_1 - t_2$.

29.18

This motivates the following weaker form of stationarity:

Defn: A random process $X(t)$ is called <u>wide-sense stationary</u> (WSS) if

(i) $E[X(t)] = \overline{X} = \text{constant}$

and

(ii) $E[X(t_1)\cdot X(t_2)] = \underbrace{R(t_1-t_2)}$

i.e., it depends on t_1 and t_2 only through the time difference $t_1 - t_2$.

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29.19

Note:

1. If a random process $X(t)$ is first and second order ($n=1,2$) stationary, and if $E[X(t)]$ and $E[X(t_1)X(t_2)]$ exist, then $X(t)$ is wide-sense stationary. The converse is <u>not</u> true.

2. If a R.P. $X(t)$ is stationary (S.S.S), it is also wide-sense stationary (W.S.S.) if $E[X(t_1)]$ and $E[X(t_1)X(t_2)]$ exist.

3. If $X(t)$ is a W.S.S R.P., $E[X(t_1)X(t)]$, $\operatorname{cov}(X(t_1), X(t_2))$, and the correlation coefficient $r_{X(t)X(t_2)}$ will depend only on the time difference $t_1 - t_2$.

29.20

Example: Consider a R.P.

$$\mathbb{X}(t) = \cos(\underset{\text{radian freq.}}{w_0} t + \Theta)$$

where Θ is a R.V. uniformly distributed on $[0, 2\pi]$:

$$f_\Theta(\theta) = \begin{cases} \frac{1}{2\pi}, & 0 \le \theta < 2\pi \\ 0, & \text{elsewhere} \end{cases}, \quad w_0 = \text{constant}$$

(radian frequency, not outcome)

Is $\mathbb{X}(t)$ W.S.S ? Let's check the two defining conditions

29.21

(i) $E[\mathbb{X}(t)] = E[\cos w_0 t + \Theta]$

$$= E[\cos(w_0 t)\cos(\Theta) - \sin(w_0 t)\sin(\Theta)]$$

$$= \int_{-\infty}^{\infty} (\cos w_0 t \cdot \cos(\theta) - \sin w_0 t \cdot \sin(\theta)) f_\Theta(\theta)\, d\theta$$

$$= \frac{\cos w_0 t}{2\pi} \underbrace{\int_0^{2\pi} \cos\theta\, d\theta}_{0} - \frac{\sin w_0 t}{2\pi} \underbrace{\int_0^{2\pi} \sin\theta\, d\theta}_{0}$$

$= 0 \quad \therefore \quad E[\mathbb{X}(t)] = 0 = \text{constant}$ ☑

n.b. Even easier

$$E[\mathbb{X}(t)] = E[\cos(w_0 t + \Theta)] = \frac{1}{2\pi} \underbrace{\int_0^{2\pi} \cos(w_0 t + \theta)\, d\theta}_{0}$$

$= 0$

integral over one period is 0.

$\cos A \cdot \cos B = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$ 29.22

$$E[X(t_1)X(t_2)]$$

$$= E\left[\cos(w_0 t_1 + \Theta) \cdot \cos(w_0 t_2 + \Theta)\right]$$

$$= E\left[\frac{1}{2}\left(\cos(w_0(t_1+t_2) + 2\Theta\right) + \frac{1}{2}\cos(w_0(t_1 - t_2))\right]$$

$$= \frac{1}{2} E\left[\cos(w_0(t_1+t_2)+2\Theta\right] + \frac{1}{2}\cos w_0(t_1-t_2) \xrightarrow{} 0$$

$$= \frac{1}{2}\cos w_0(t_1 - t_2) \quad \boxtimes$$

$\therefore$ (i) $E[X(t)] = 0 =$ constant

(ii) $E[X(t_1)X(t_2)] = \frac{1}{2}\cos w_0(t_1 - t_2) =$ a funtion of t_1 and t_2 through the difference $t_1 - t$

$\therefore$ $X(t)$ is wide sense stationary.