Session 26

Convergence of Sequences of RVs

26.1

$$X_1, X_2, \ldots, X_n, \ldots$$

Example: Suppose I make a sequence of measurements:

$$X_k = a + W_k, \quad k = 1, 2, 3, \ldots$$

a = parameter of interest.

X_k = k-th measurement.

W_k = experimental error in the k-th measurement

$E[W_k] = 0$ (unbiased measurement)

26.2

If we consider $X_1, \ldots, X_n \ldots$ to be measurements, we typically estimate the value of a by

$$Y_n = \frac{1}{n}\left[X_1 + X_2 + \cdots + X_n\right]$$

Is this a good estimate of a?

Hopefully, $Y_n \longrightarrow a$, as $n \to \infty$.

Is this true? Is it always true?

Is it ever true?

What does $Y_n \longrightarrow a$ mean?

26.3

Note that

$$Y_n = \frac{1}{n}\left[X_1 + \cdots + X_n\right]$$

is itself a random variable, so

$$Y_1, Y_2, Y_3, \ldots, Y_n, \ldots$$

is also a sequence of random variables (i.e., a random sequence.)

<u>Q</u>: What does it mean to ask if

$$Y_n \longrightarrow a \text{ as } n \longrightarrow \infty \text{ ?}$$

Defn: A random sequence or a discrete-time stochastic process is a sequence of random variables $\mathbb{X}_1, \ldots, \mathbb{X}_n, \ldots$ defined on $(\mathcal{S}, \mathcal{F}, P)$. 26.4

(If $\mathbb{X}_i$ is a R.V. for each $i \in \mathbb{N}$, then measurability is not an issue.)

- We often write a random sequence as $\{\mathbb{X}_n\}$ or $\{\mathbb{X}_n\}_{n \in \mathbb{N}}$ or $\{\mathbb{X}_n\}_{n \geq 1}$
- For any specific $\omega_0 \in \mathcal{S}$ of $(\mathcal{S}, \mathcal{F}, P)$, $\mathbb{X}_1(\omega_0), \mathbb{X}_2(\omega_0), \ldots, \mathbb{X}_n(\omega_0), \ldots$ is a sequence of real numbers.

Defn: A sequence of real numbers $x_1, \ldots, x_n, \ldots$ is said to converge to a limit x if, for $\forall \varepsilon > 0$, there exists a number $n_\varepsilon \in \mathbb{N}$ such that 26.5

$$|x_n - x| < \varepsilon, \quad \forall n \geq n_\varepsilon.$$

"$x_n \to x$ as $n \to \infty$"

26.6

Given a random sequence

$X_1(\cdot), X_2(\cdot), \dots, X_n(\cdot), \dots$

for any particular $\in \Omega$, we have

$X_1(\omega), X_2(\omega), \dots, X_n(\omega), \dots$

is a sequence of real numbers

- It may converge to a number $X(\omega)$
- or, it may not converge.

n.b. The $X(\omega)$ that the random sequence converges to is itself a function of ω ($X(\omega)$ is a R.V.).

26.7

Given a random sequence

$X_1(\cdot), X_2(\cdot), \dots, X_n(\cdot), \dots$

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- It may converge to a number $X(\omega)$
- or, it may not converge.

n.b. The $X(\omega)$ that the random sequence converges to is itself a function of ω ($X(\omega)$ is a R.V.).

Given a random sequence

$\mathbb{X}_1(\omega), \ldots, \mathbb{X}_n(\omega), \ldots$

most likely it will converge for some $\omega \in \mathcal{S}$, and not converge for other $\omega \in \mathcal{S}$.

When we study convergence of random sequences (Stochastic Convergence) we study The set $A \subset \mathcal{S}$ for which

$\mathbb{X}_1(\omega), \ldots, \mathbb{X}_n(\omega), \ldots$

is a convergent sequence for all $\omega \in A$.

<u>Defn</u>: We say that a sequence of RVs <u>converges everywhere</u> (e) if the sequences

$\mathbb{X}_1(\omega), \mathbb{X}_2(\omega), \ldots, \mathbb{X}_n(\omega), \ldots$

each converge to a number $\mathbb{X}(\omega)$ for each $\omega \in \mathcal{S}$.

<u>n.b.</u>
- The number $\mathbb{X}(\omega)$ that each sequence converges to for each $\omega \in \mathcal{S}$ is a function of ω.
  $\mathbb{X}(\omega)$ is a RV.
- Convergence everywhere is too strong or restrictive to be useful.

26.10

Defn: A random sequence $\{X_n\}$ converges almost everywhere (a.e.) if the set of outcomes $A \subset S$ such that $X_n(\omega) \xrightarrow{n\to\infty} X(\omega)$, $\omega \in A$ exists and has probability 1:

$$P(A) = 1.$$

Other names for convergence almost everywhere.

- also sure convergence (a.s.)
- convergence with probability one.

we write this as

$$\text{“} X_n \xrightarrow{a.e} X \text{”}$$

$$\text{“} P(\{X_n \to X\}) = 1 \text{”}$$

26.11

Defn: A random sequence $\{X_n\}$ converges in mean-square (m.s) to a RV X if

$$E[|X_n - X|^2] \to 0 \text{ as } n \to \infty.$$

n.b. If we have the j-pdf of X_n and X, we can compute

$$E[|X_n - X|^2] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x_n - x)^2 f_{X_n X}(x_n, x)\, dx_n\, dx$$

So in principle, it is easy to determine mean-square convergence

Defn: A random sequence $\{X_n\}$ <u>converges in probability</u> (p) to a random variable X if, $\forall \varepsilon > 0$

$$P(\{|X_n - X| > \varepsilon\}) \to 0 \text{ as } n \to \infty.$$

Defn: A random sequence $\{X_n\}$ <u>converges in distribution</u> (d) to a RV X if

$$F_{X_n}(x) \longrightarrow F_X(x) \text{ as } n \to \infty \quad (*)$$

at every point $x \in \mathbb{R}$ where $F_X(x)$ is continuous.

(*): i.e. $\forall \varepsilon > 0$, $\exists n_\varepsilon \in \mathbb{N}$ such that

$$|F_{X_n}(x) - F_X(x)| < \varepsilon, \ \forall n \geq n_\varepsilon$$

for all $x \in \mathbb{R}$ where $F_X(x)$ is continuous.

Defn: A random sequence $\{X_n\}$ converges in density (den) to a R.V. X if

$$f_{X_n}(x) \longrightarrow f_X(x) \text{ as } n \to \infty$$

at every point $x \in \mathbb{R}$ where $F_X(x)$ is continuous.

Aren't convergence in density and convergence in distribution equivalent?

No!

Example: Let $\{X_n\}$ be a sequence of RVs having p.d.fs

$$f_{X_n}(x) = [1 + \cos(2\pi n x)] \cdot 1_{[0,1]}(x).$$

n.b. (i) $f_{X_n}(x) \geq 0, \forall x$

(ii) $= \int_{-\infty}^{\infty} f_n(x)\,dx = \int_0^1 (1 + \cos(2\pi n x))\,dx = \left(x + \frac{1}{2\pi n}\sin(2\pi n x)\right)\Big|_0^1$
$= 1, \; n = 1, 2, 3, \ldots$

26.16

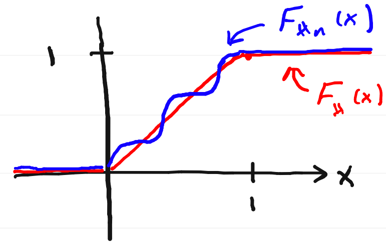
Now define

$$F_{*}(x) = \begin{cases} 0, & x < 0, \\ x, & x \in [0,1], \\ 1, & x > 1. \end{cases}$$

Note that

$$F_{*_n}(x) = \int_{-\infty}^{x} f_n(\alpha)\, d\alpha = \begin{cases} 0, & x < 0 \\ x + \frac{1}{2\pi n} \sin(2\pi n x), & x \in [0,1] \\ 1, & x > 1 \end{cases}$$

Clearly $F_{*_n}(x) \longrightarrow F_{*}(x), \ \forall x \in \mathbb{R}$



$F_{*_n}(x) \to F_{*}(x)$ as $n \to \infty$.

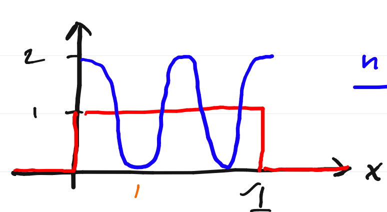
26.17

But $f_{*_n}(x) \not\to f_{*}(x)$

$$f_{*}(x) = \frac{d F_{*}(x)}{dx} = \mathbb{1}_{[0,1]}(x)$$

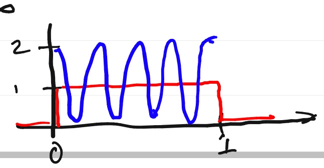
and

$$f_n(x) = [1 + \cos 2\pi n x] \cdot \mathbb{1}_{[0,1]}(x)$$



$\underline{n = 2}$

As $n \to \infty$



$f_{*_n}(x) \not\to f_{*}(x)$ as $n \to \infty$.

$\therefore$ Convergence $(d) \not\Rightarrow$ convergence (den).

26.18

However

$$\text{convergence (den)} \not\Rightarrow \text{convergence (d)}.$$

(n.b. The integration that converts $f_{X_n}(x)$ into $F_{X_n}(x)$ smoothes things out.)

Cauchy Criterion for Convergence

26.19

Recall that a sequence of real numbers $x_1, \ldots, x_n, \ldots$ converges to a limit x if $\forall \varepsilon > 0$, $\exists n_\varepsilon \in \mathbb{N}$ such that

$$|x_n - x| < \varepsilon, \quad \forall n \geq n_\varepsilon.$$

To use this definition to determine if $x_1, \ldots, x_n, \ldots$ converges, we have to know the limit x.

The Cauchy Criterion gives us a way to test for convergence without knowing the limit x.

Cauchy Criterion: If $\{x_n\}$ is a sequence of real numbers and

$$|x_{n+m} - x_n| \to 0 \quad \text{as } n \to \infty$$

for all $m \in \mathbb{N}$, then the sequence converges to a real number.

n.b. The Cauchy criterion can be applied to various forms of stochastic convergence.

e.g. $E[|X_{n+m} - X_n|^2] \to 0$ as $n \to \infty$ for all $m = 1, 2, 3, 4 \ldots$ then $\{X_n\}$ converges in mean-square.

Comparison of Modes of Convergence