Session 25

Recall

Random Vectors

25.1

- CDFs
- PDFs
- Marginal pdfs and cdfs
- Transformations of RVecs < "Fundamental", Direct

- We've considered two RVs on $(\Omega, \mathcal{F}, P)$.
- We can extend this to n RVs on $(\Omega, \mathcal{F}, P)$:

$$\mathbb{X}_1(\omega), \mathbb{X}_2(\omega), \ldots, \mathbb{X}_n(\omega).$$

- We can arrange these RVs as elements of a vector.

(A <u>random vector</u>.)

Statistical Independence of Multiple Random Variables

Defn: The random variables $X_1, \ldots, X_n$ are statistically independent iff all events of the form

$\{X_1 \in A_1\}, \{X_2 \in A_2\}, \ldots, \{X_n \in A_n\}$

are statistically independent, where

$A_1 \in B(\mathbb{R}), A_2 \in B(\mathbb{R}), \ldots, A_n \in B(\mathbb{R})$,

for all Borel sets $A_1, \ldots, A_n$.

Equivalently,

Defn': The jointly distributed random variables $X_1, X_2, \ldots, X_n$ are statistically independent iff

$$f_{X_1, \ldots, X_n}(x_1, \ldots, x_n) = f_{X_1}(x_1) f_{X_2}(x_2) \cdots f_{X_n}(x_n).$$

25.4

Conditional Densities, Characteristic functions and Normality (n-dim Gaussian)

Conditional densities involving $\mathbb{X}_1, \ldots, \mathbb{X}_n$ generalize easily from the 2-dim case:

$$f(x_1, \ldots, x_k \mid x_{k+1}, \ldots, x_n) = \frac{f(x_1, \ldots, x_n)}{f(x_{k+1}, \ldots, x_n)}$$

(n.b. $f(x_1, x_3 \mid x_2, x_4) = \dfrac{f(x_1, x_2, x_3, x_4)}{f(x_2, x_4)}$)

25.5

Recall that the correlation between two RVs $\mathbb{X}_j$ and $\mathbb{X}_k$ is

$$R_{jk} = E[\mathbb{X}_j \mathbb{X}_k]$$

and their covariance is given by

$$C_{jk} = E\left[(\mathbb{X}_j - \overline{\mathbb{X}}_j)(\mathbb{X}_k - \overline{\mathbb{X}}_k) \right]$$

We are interested in tabulating these values for the RVec

$$\underline{\mathbb{X}} = (\mathbb{X}_1, \ldots, \mathbb{X}_n)$$

25.6

Defn: Given a RVec $\underline{X} = (X_1, \dots, X_n)$ we define the correlation matrix

$$R_{\underline{X}} = \begin{pmatrix} R_{11} & R_{12} & \cdots & R_{1n} \\ R_{21} & R_{22} & \cdots & R_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ R_{n1} & R_{n2} & \cdots & R_{nn} \end{pmatrix}$$

and the covariance matrix

$$C_{\underline{X}} = \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{2} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{pmatrix}$$

25.7

If X_j and X_k are complex RVs, then

$$R_{jk} \triangleq E[X_j X_k^*]$$

$$C_{jk} \triangleq E[(X_j - \overline{X}_j)(X_k - \overline{X}_k)^*]$$

n.b. A complex RV Z has the form

$$Z = X + iY,$$

where X and Y are j-dist real RVs.

$$Z^* = X - iY$$

25.8

n.b. $\underline{X} = (X_1, \ldots, X_n)$ ~ row vector

$$E[\underline{X}] \triangleq (E[X_1], \ldots, E[X_n]) = \overline{\underline{X}}.$$

We can also write

$$R_{\underline{X}} = E[\underline{X}^T \underline{X}^*]$$

and

$$C_{\underline{X}} = E[(\underline{X} - \overline{\underline{X}})^T (\underline{X} - \overline{\underline{X}})^*]$$

n.b. For a row vector $\underline{x} = (x_1, \ldots, x_n)$

$$\underline{x}^T \underline{x} = \begin{pmatrix} x_1x_1 & x_1x_2 & \cdots & x_1x_n \\ x_2x_1 & x_2x_2 & \cdots & x_2x_n \\ \vdots & \vdots & & \vdots \\ x_nx_1 & x_nx_2 & \cdots & x_nx_n \end{pmatrix} \quad \text{(outer product)}$$

$$\underline{x}\,\underline{x}^T = x_1x_1 + x_2x_2 + \cdots + x_nx_n \quad \text{(inner product)}$$

24.7

Opposite of what we are used to for column vectors.

25.9

Defn: An $n \times n$ matrix B is said to be non-negative definite if

$$\sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j^* b_{ij} \geq 0$$

for all complex numbers $x_1, \ldots, x_n$, where

$$B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix}.$$

If the inequality is strict (>0) for all $x_1, \ldots, x_n$ not all zero, then we say that the matrix B is positive definite.

25.10

Theorem: The correlation matrix $R_{\underline{X}}$ of any RVec $\underline{X}$ is non-negative definite.

Proof:

$$0 \leq E\left[|a_1 X_1 + a_2 X_2 + \cdots a_n X_n|^2 \right]$$

$$= E\left[\sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j^* X_i X_j^* \right]$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j^* E[X_i X_j^*]$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j^* R_{ij}, \quad \text{for all } a_1, \ldots, a_n \in \mathbb{C}$$

$\Rightarrow R_{\underline{X}}$ is non-negative definite ∎

Corollary: $\underline{C_X}$ is non-negative definite.

n.b. $C_{\underline{X}} = R_{\underline{\tilde{X}}}$, $\underline{\tilde{X}_1} = \underline{X_1} - \underline{\overline{X}_1}$, ..., $\underline{\tilde{X}_n} = \underline{X_n} - \underline{\overline{X}_n}$.

25.11

Char. Ftn. of a RVec

Let $\underline{X} = (X_1, \ldots, X_n)$ be a RVec. Then the Characteristic function of $\underline{X}$ is

$$\Phi_{\underline{X}}(\omega_1, \ldots, \omega_n) = \Phi_{\underline{X}}(\underline{\Omega})$$

$$\triangleq E\left[e^{i(\omega_1 X_1 + \omega_2 X_2 + \cdots + \omega_n X_n)} \right]$$

$$= E\left[e^{i \underline{\Omega}\, \underline{X}^T} \right]$$

where

$$\underline{\Omega} = (\omega_1, \ldots, \omega_n).$$

n.b. Let $Z = X_1 + X_2 + \cdots + X_n$ 25.12

Then if $\Phi_{\underline{X}}(\omega_1, \ldots, \omega_n)$ is the char. fth. of $X_1, \ldots, X_n$, then

$$\Phi_Z(\omega) = E[e^{i\omega Z}] = E[e^{i\omega(X_1 + X_2 + \cdots + X_n)}]$$
$$= E[\,{}^{i(\omega X_1 + \omega X_2 + \cdots + \omega X_n)}]$$
$$= \Phi_{\underline{X}}(\omega, \omega, \cdots, \omega)$$

Furthermore, if $X_1, \ldots, X_n$ are stat. indep., then

$$\Phi_{X_1, \ldots, X_n}(\omega_1, \ldots, \omega_n) = \Phi_{X_1}(\omega_1) \cdot \Phi_{X_2}(\omega_2) \cdots \Phi_{X_n}(\omega_n)$$

and thus

$$\Phi_Z(\omega) = \Phi_{X_1}(\omega) \cdot \Phi_{X_2}(\omega) \cdots \Phi_{X_n}(\omega).$$

Gaussian RVecs and their Char. Ftns. 25.13

Defn: Let $X_1, \ldots, X_n$ be a RVec of dimension n. Then $\underline{X} = (X_1, \ldots, X_n)$ is a Gaussian RVec, and $X_1, \ldots, X_n$ are jointly Gaussian iff

$$Z = a_0 + \sum_{j=1}^{n} a_j X_j$$

is a Gaussian RV for all $a_0, a_1, \ldots, a_n \in \mathbb{R}$.

25.14

The joint characteristic function of such a Gaussian RVec has the following form:

$$\underline{\Phi}_{\underline{X}}(\underline{\Omega}) = \Phi_{X_1,\dots,X_n}(\omega_1,\dots,\omega_n)$$

$$= e^{i\underline{\Omega}\,\underline{m}_{\underline{X}}^T} e^{-\frac{1}{2}\underline{\Omega}\, C_{\underline{X}}\, \underline{\Omega}^T}$$

$$= e^{i\sum_{j=1}^{n}\omega_j\mu_j} e^{-\frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\omega_j\omega_k c_{jk}} \qquad (*)$$

where

$$\underline{m}_{\underline{X}} = (E[X_1],\dots,E[X_n]) = (\mu_1,\dots,\mu_n)$$

and

$$C_{\underline{X}} = E[(\underline{X}-\underline{m}_{\underline{X}})^T(\underline{X}-\underline{m}_{\underline{X}})]$$

$$= [c_{jk}] \qquad n\times n \text{ covariance matrix of } \underline{X}$$

25.15

n.b. Suppose that $X_1,\dots,X_n$ are j-Gaussian with char. fcn (*)

Let $Z = \sum_{k=1}^{n} a_k X_k$. What is distribution of Z?

Show that Z is Gaussian and find its mean and variance (exercise)

$$\Phi_Z(\omega) = e^{i\omega\mu_Z} e^{-\frac{1}{2}\omega^2\sigma_Z^2}$$

So we have shown the following important result:

<u>Theorem:</u> Let $X_1, X_2, \ldots, X_n$ be jointly distributed, jointly Gaussian random variables. Then any linear combination

$$Z = \sum_{j=1}^{n} a_j X_j$$

is a Gaussian random variable, for any $a_1, a_2, \ldots, a_n \in \mathbb{R}$.

Convergence of Sequences of RVs

$$X_1, X_2, \ldots, X_n, \ldots$$

<u>Example:</u> Suppose I make a sequence of measurements:

$$X_k = a + W_k, \quad k = 1, 2, 3, \ldots$$

a = parameter of interest.

X_k = k-th measurement

W_k = experimental error of k-th measurement

$E[W_k] = 0$ (unbiased measurement)

We consider $\mathbb{X}_1, \dots, \mathbb{X}_n, \dots$ to be measurements, and we typically estimate the value of a by

$$\mathbb{Y}_n = \frac{1}{n}\left[\mathbb{X}_1 + \mathbb{X}_2 + \cdots + \mathbb{X}_n\right]$$

Is this a good estimate of a?

Hopefully, $\mathbb{Y}_n \longrightarrow a$, as $n \to \infty$.

Is this true? Is it always true?

Is it ever true?

What does it mean?