Session 23

Joint Gaussian Examples:

23.1

Recall:

- We have seen that

$$f_X(x \mid \{Y = y\}) = \frac{f_{XY}(x,y)}{f_Y(y)}$$

- We know that

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx$$

and $f_{XY}(x,y) = f_Y(y \mid X = x)\, f_X(x)$

$$\Rightarrow \quad f_X(x \mid \{Y = y\}) = \frac{f_Y(y \mid \{X = x\})\, f_X(x)}{\int_{-\infty}^{\infty} f_Y(y \mid \{X = \alpha\})\, f_X(\alpha)\, d\alpha}$$

(Bayes Theorem)

23.2

Also recall: We investigated two estimators for estimating the value of $\mathbb{X}$ given $\{\mathbb{Y}=y\}$:

$$\hat{\mathbb{X}}_{MMS}(y) = E\left[\mathbb{X} \mid \{\mathbb{Y}=y\}\right],$$

and

$$\hat{\mathbb{X}}_{MAP}(y) = \arg\max_{x} \left\{ f_{\mathbb{X}}(x \mid \{\mathbb{Y}=y\}) \right\}.$$

Both of these estimators require that we find

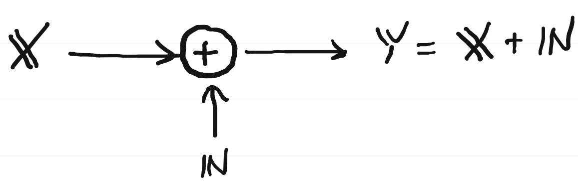
$$f_{\mathbb{X}}(x \mid \{\mathbb{Y}=y\}).$$

23.3

Example: Let $\mathbb{X}$ and $\mathbb{N}$ be two zero-mean, jointly distributed independent Gaussian RVs, with variances σ_X^2 and σ_N^2, respectively.

$\mathbb{X} \perp\!\!\!\perp \mathbb{N}$

Now consider a new RV $\mathbb{Y}$:



Suppose I observe $\mathbb{Y}=y$. What is the Minimum Mean-Square Error estimator of $\mathbb{X}$ given $\{\mathbb{Y}=y\}$?

23.4

$$\mathbb{X} = \mathbb{X}$$
$$\mathbb{Y} = \mathbb{X} + \mathbb{N}$$

$$\Phi_{\mathbb{X}\mathbb{Y}}(\omega_1,\omega_2) = E\left[e^{i(\omega_1\mathbb{X}+\omega_2\mathbb{Y})}\right]$$
$$= E\left[e^{i(\omega_1\mathbb{X} + \omega_2(\mathbb{X}+\mathbb{N}))}\right]$$
$$= E\left[e^{i((\omega_1+\omega_2)\mathbb{X} + \omega_2\mathbb{N})}\right]$$
$$= E\left[e^{i(\omega_1+\omega_2)\mathbb{X}}\right]\cdot E\left[e^{i\omega_2\mathbb{N}}\right]$$
$$= \Phi_{\mathbb{X}}(\omega_1+\omega_2)\cdot\Phi_{\mathbb{N}}(\omega_2)$$

$$\Phi_{\mathbb{X}\mathbb{Y}}(\omega_1,\omega_2) = e^{-\frac{1}{2}(\omega_1^2\sigma_x^2 + 2r\sigma_1\sigma_2\sigma_x\sigma_y + \omega_2^2\sigma_y^2)}$$

23.5

$$\hat{x}_{MMS}(y) = E[\mathbb{X} \mid \{\mathbb{Y}=y\}]$$

So I need to find $f_{\mathbb{X}}(x \mid \{\mathbb{Y}=y\}) = \dfrac{f_{\mathbb{X}\mathbb{Y}}(x,y)}{f_{\mathbb{Y}}(y)}$

We need to find the j-pdf $f_{\mathbb{X}\mathbb{Y}}(x,y)$ of $\mathbb{X}$ and $\mathbb{Y}$:

- Easy to show $\sigma_Y^2 = \sigma_X^2 + \sigma_N^2$.
- Also, $r_{XY} = \dfrac{E[\mathbb{X}\mathbb{Y}] - E[\mathbb{X}]\cdot E[\mathbb{Y}]}{\sigma_X\sigma_Y}$

(exercise)

$$= \cdots = \sqrt{\frac{\sigma_X^2}{\sigma_X^2+\sigma_N^2}}$$

Aside 1:

$$r_{XY} = \frac{\operatorname{cov}(X,Y)}{\sigma_X \sigma_Y}$$

$$\operatorname{var}(Y) = \sigma_X^2 + \sigma_N^2, \quad \sigma_Y = \sqrt{\sigma_X^2 + \sigma_N^2}$$

n.b. $\operatorname{Cov}(X,Y) = E[XY] - E[X]\cdot E[Y]$

$$E[XY] = E[X(X+N)] = E[X^2 + X\cdot N]$$
$$= E[X^2] + E[X\cdot N]$$
$$= \sigma_X^2 + \cancel{E[X]}^{0}\cdot \cancel{E[N]}^{0}$$
$$= \sigma_X^2$$
$$\Rightarrow \operatorname{cov}(X,Y) = \sigma_X^2$$

$$r_{XY} = \frac{\operatorname{cov}(X,Y)}{\sigma_X \sigma_Y} = \frac{\sigma_X^2}{\sigma_X \sigma_Y} = \frac{\sigma_X}{\sigma_Y} = \sqrt{\frac{\sigma_X^2}{\sigma_X^2 + \sigma_N^2}}$$

$$f_X(x \mid \{Y = y\}) = \frac{f_{XY}(x,y)}{f_Y(y)}$$

$$= \frac{\dfrac{1}{2\pi \sigma_X \sigma_Y \sqrt{1-r^2}} \exp\left\{-\dfrac{1}{2(1-r^2)}\left[\dfrac{x^2}{\sigma_X^2} - \dfrac{2rxy}{\sigma_X\sigma_Y} + \dfrac{y^2}{\sigma_Y^2}\right]\right\}}{\dfrac{1}{\sqrt{2\pi}\,\sigma_Y} \exp\left\{\dfrac{-y^2}{2\sigma_Y^2}\right\}}$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_X\sqrt{1-r^2}} \exp\left\{\frac{(-)1}{2(1-r^2)}\left[\frac{x^2}{\sigma_X^2} - \frac{2rxy}{\sigma_X\sigma_Y} + \frac{y^2}{\sigma_Y^2} - (1-r^2)\frac{y^2}{\sigma_Y^2}\right]\right\}$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_X\sqrt{1-r^2}} \exp\left\{-\frac{1}{2(1-r^2)}\left[\frac{x^2}{\sigma_X^2} - \frac{2rxy}{\sigma_X\sigma_Y} + \frac{r^2y^2}{\sigma_Y^2}\right]\right\}$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_X\sqrt{1-r^2}} \exp\left\{\frac{-1}{2(1-r^2)}\left[\frac{x}{\sigma_X} - \frac{ry}{\sigma_Y}\right]^{2}\right\}$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_X\sqrt{1-r^2}} \exp\left\{\frac{-1}{2(1-r^2)\sigma_X^2}\left[x - r\frac{\sigma_X}{\sigma_Y}y\right]^2\right\}$$

23.8

So we see here that

$$f_{X}(x|Y=y) = \frac{1}{\sqrt{2\pi}\,\sigma_x\sqrt{1-r^2}} \exp\left\{\frac{-1}{2\sigma_x^2(1-r^2)}\left[x-r\frac{\sigma_x}{\sigma_Y}y\right]^2\right\},$$

which is a Gaussian pdf with mean $r\frac{\sigma_X}{\sigma_Y}y$ and variance $\sigma_x^2(1-r^2)$. So by inspection, we have

$$\hat{X}_{MMS}(y) = r\frac{\sigma_X}{\sigma_Y} = \sqrt{\frac{\sigma_x^2}{\sigma_x^2+\sigma_N^2}}\cdot\sqrt{\frac{\sigma_x^2}{\sigma_x^2+\sigma_N^2}}\cdot y$$

$$= \left(\frac{\sigma_x^2}{\sigma_x^2+\sigma_N^2}\right)y \quad = \hat{X}_{MAP}(y)$$

peak of Gaussian occurs at mean.

23.9

Also, it can be shown (exercise) that

$$f_{X}(x|\{Y=y\}) = \frac{f_{XY}(x,y)}{f_Y(y)}$$

(exercise)

$$= \cdots = \frac{1}{\sqrt{2\pi}\,\sigma_x\sqrt{1-r_{XY}^2}} \exp\left\{\frac{-(x-r_{XY}\frac{\sigma_X}{\sigma_Y}y)^2}{2\sigma_x^2(1-r_{XY}^2)}\right\}$$

$$\therefore \hat{X}_{MMS}(y) = E[X|\{Y=y\}] = r_{XY}\frac{\sigma_X}{\sigma_Y}y$$

$$= \cdots = \left(\frac{\sigma_x^2}{\sigma_x^2+\sigma_N^2}\right)y$$

Exam 2 Cut-off

24.25

Recall

Random Vectors

23.10

- We've considered two RVs on $(\Omega, \mathcal{F}, P)$.
- We can extend this to n RVs on $(\Omega, \mathcal{F}, P)$:

$$\mathbb{X}_1(\omega), \mathbb{X}_2(\omega), \dots, \mathbb{X}_n(\omega).$$

- We can arrange these RVs as elements of a vector.

(A random vector.)

23.11

Defn: Let $X_1, \ldots, X_n$ be n jointly distributed RVs on $(S, \mathcal{F}, P)$. Then the vector of RVs

$$\underline{X} = (X_1, \ldots, X_n) \leftarrow \text{Row Vector}$$

is a random vector (RVec) defined on $(S, \mathcal{F}, P)$.

Alternatively, we can think of a RVec as a mapping from S to $\mathbb{R}^n$

$$\underline{X} : S \to \mathbb{R}^n$$

23.12

CDF: $$F_{\underline{X}}(\underline{x}) = F_{\underline{X}}(x_1, \ldots, x_n)$$

$$= F_{X_1 \ldots X_n}(x_1, \ldots, x_n)$$

$$= P(\{X_1 \leq x_1\} \cap \{X_2 \leq x_2\} \cap \cdots \cap \{X_n \leq x_n\})$$

PDF: $$f_{\underline{X}}(\underline{x}) = f_{\underline{X}}(x_1, \ldots, x_n)$$

$$= f_{X_1 \ldots X_n}(x_1, \ldots, x_n)$$

$$= \frac{\partial^n F_{\underline{X}}(x_1, \ldots, x_n)}{\partial x_1 \, \partial x_2 \cdots \partial x_n}$$

23.13

Let $D \subset \mathbb{R}^n$ $(D \in \mathcal{B}(\mathbb{R}^n))$

Then $P(\{\underline{X} \in D\}) = \int_D f_{\underline{X}}(\underline{x})\, d\underline{x}$

$$= \int_{\mathbb{R}^n} f_{\underline{X}}(\underline{x}) \cdot \mathbb{1}_D(\underline{x})\, d\underline{x}$$

$$= \underbrace{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty}}_{\substack{n\text{-fold} \\ \text{integration}}} f_{\underline{X}}(x_1, \dots, x_n) \cdot \mathbb{1}_D((x_1, \dots, x_n))\, dx_1 \cdots dx_n$$

23.14

All the general properties of cdfs and pdfs generalize from 2-dimensions to n-dimensions:

e.g. Given j-dist RVs X_1, X_2, X_3, X_4

$$F_{X_1,X_3}(x_1, x_3) = F_{X_1 X_2 X_3 X_4}(x_1, +\infty, x_3, +\infty).$$

or

$$f_{X_1,X_3}(x_1, x_3) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X_1 X_2 X_3 X_4}(x_1, x_2, x_3, x_4)\, dx_2\, dx_4$$

Transformations on RVecs

Given $\underline{X} = (X_1, \ldots, X_n) \sim$ RVec

we can define a new RVec

$$\underline{Y} = (Y_1, \ldots, Y_k)$$

where $Y_j = g_j(\underline{X})$, $j = 1, \ldots, k$

where $k \gtreqless n$

How do we find $F_{\underline{Y}}(y)$ or $f_{\underline{Y}}(y)$?

Given $\underline{X} = (X_1, \ldots, X_n) \sim$ RVec

we can define the new RVec

$$\underline{Y} = (Y_1, \ldots, Y_k),$$

where

$$Y_j = g_j(\underline{X}), \quad j = 1, \ldots, k.$$

Here $k \gtreqless n$.

How do we find $F_{\underline{Y}}(y)$ or $f_{\underline{Y}}(y)$?

23.17

To find $F_{\underline{Y}}(y_1, \ldots, y_k)$, define

$$D(y_1, \ldots, y_k) \triangleq \{(x_1, \ldots, x_n) \in \mathbb{R}^n : g_1(x_1, \ldots, x_n) \le y_1, \ldots, g_k(x_1, \ldots, x_n) \le y_k\} \subset \mathbb{R}^n \ (\in \mathcal{B}(\mathbb{R}^n))$$

Then

$$F_{\underline{Y}}(y_1, \ldots, y_k) = \int \cdots \int_{D(y_1, \ldots, y_k)} f_{\underline{X}}(x_1, \ldots, x_n)\, dx_1 \cdots dx_n$$

and

$$f_{\underline{Y}}(y_1, \ldots, y_k) = \frac{\partial^k F_{\underline{Y}}(y_1, \ldots, y_k)}{\partial y_1\, \partial y_2 \cdots \partial y_k}.$$

23.18

Direct Approach (k = n and one-to-one mapping)

Theorem: Given RVec $\underline{X} = (X_1, \ldots, X_n)$, define a new RVec $\underline{Y} = (Y_1, \ldots, Y_n)$ by the one-to-one mapping

$$G : \mathbb{R}^n \to \mathbb{R}^n$$

described by

$$Y_1 = g_1(\underline{X}), \ \ldots, \ Y_n = g_n(\underline{X}).$$

Let $X_1 = h_1(\underline{Y}) = x_1(\underline{Y}), \ldots,$

$X_n = h_n(\underline{Y}) = x_n(\underline{Y})$.

Then the j-pdf of $\underline{Y}$ is

$$f_{\underline{Y}}(y_1, \dots, y_n) = f_{\underline{X}}(x_1(\underline{y}), \dots, x_n(\underline{y})) \left| \frac{\partial(x_1, \dots, x_n)}{\partial(y_1, \dots, y_n)} \right|$$

where the Jacobian is given by

$$\frac{\partial(x_1, \dots, x_n)}{\partial(y_1, \dots, y_n)} = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \cdots & \frac{\partial x_1}{\partial y_n} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} & \cdots & \frac{\partial x_2}{\partial y_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial y_1} & \frac{\partial x_n}{\partial y_2} & \cdots & \frac{\partial x_n}{\partial y_n} \end{vmatrix}$$

<u>n.b.</u> If you have $Y_1, \dots, Y_k$, where $k < n$, you can introduce aux. variables.