Session 21

## Joint Characteristic Functions

21.1

**Defn:** The joint characteristic function of two j-dist RVs $X$ and $Y$ is

$$\Phi_{XY}(\omega_1, \omega_2) \triangleq E\left[e^{i(\omega_1 X + \omega_2 Y)}\right]$$

$$= \underbrace{\iint_{\mathbb{R}^2} e^{+i(\omega_1 x + \omega_2 y)} f_{XY}(x,y)\, dx\, dy}_{\text{2-Dim Fourier Transform}}$$

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n.b. Inverse Fourier Transform Relationship:

$$f_{XY}(x,y) = \frac{1}{(2\pi)^2} \iint_{\mathbb{R}^2} \Phi_{XY}(\omega_1,\omega_2)\, e^{-i(\omega_1 x + \omega_2 y)}\, d\omega_1\, d\omega_2$$

Note: 1. $\Phi_X(\omega) = \Phi_{XY}(\omega, 0)$

$\Phi_Y(\omega) = \Phi_{XY}(0, \omega)$

2. $Z = aX + bY$

$$\Phi_Z(\omega) = E[e^{i\omega Z}] = E[e^{i\omega(aX+bY)}]$$

$$= E[e^{i[(a\omega)X + (b\omega)Y]}]$$

$$= \Phi_{XY}(a\omega, b\omega)$$

21.3

**Theorem** ("Convolution Theorem")

Let $X$ and $Y$ be two j-dist statistically independent random variables, and let $Z = X + Y$. Then

$$\Phi_Z(\omega) = \Phi_X(\omega) \cdot \Phi_Y(\omega).$$

Proof: $\Phi_Z(\omega) = E[e^{i\omega Z}] = E[e^{i\omega(X+Y)}]$

$= E[e^{i\omega X} \cdot e^{i\omega Y}]$ - - - (*)

$= E[e^{i\omega X}] \cdot E[e^{i\omega Y}]$

$= \Phi_X(\omega) \cdot \Phi_Y(\omega)$ ∎

(*): n.b.

$$E\left[e^{i\omega X}\cdot e^{i\omega Y}\right] = \iint_{\mathbb{R}^2} e^{i\omega x}\cdot e^{i\omega y} f_{XY}(x,y)\,dx\,dy$$

$\Downarrow$ $X \perp\!\!\!\perp Y$

$$= \iint_{\mathbb{R}^2} e^{i\omega x}\cdot e^{i\omega y} f_X(x)\cdot f_Y(y)\,dx\,dy$$

$$= \int_{\mathbb{R}} e^{i\omega x} f_X(x)\,dx \cdot \int_{\mathbb{R}} e^{i\omega y} f_Y(y)\,dy$$

$$= E\left[e^{i\omega X}\right]\cdot E\left[e^{i\omega Y}\right]$$



**Fact:** The joint characteristic function of two jointly Gaussian random variables $X$ and $Y$ with j-pdf

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-r^2}}\exp\left\{\frac{-1}{2(1-r^2)}\left[\frac{(x-\mu_X)^2}{\sigma_X^2} - 2r\frac{(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y} + \frac{(y-\mu_Y)^2}{\sigma_Y^2}\right]\right\}$$

is

$$\boxed{\Phi_{XY}(\omega_1,\omega_2) = e^{i(\mu_X\omega_1+\mu_Y\omega_2)}\, e^{-\frac{1}{2}\left[\sigma_X^2\omega_1^2 + 2r\sigma_X\sigma_Y\omega_1\omega_2 + \sigma_Y^2\omega_2^2\right]}.}$$

Memorize this!

**Proof:** ( You could prove this using the definition — 2D Fourier transform of a 2D Gaussian.)

The following approach is easier:

- It can be shown that if

$$Z = w_1 X + w_2 Y \quad , \quad w_1, w_2 \in \mathbb{R}$$

and $X$ and $Y$ are jointly Gaussian, then $Z$ is a Gaussian RV.

- If $Z$ is a Gaussian RV, then it has characteristic function

$$\Phi_Z(\omega) = e^{i \mu_Z \omega} e^{-\frac{1}{2} \sigma_Z^2 \omega^2}$$

where

$$\mu_Z = w_1 \mu_X + w_2 \mu_Y$$

and

$$\sigma_Z^2 = w_1^2 \sigma_X^2 + 2 r w_1 w_2 \sigma_X \sigma_Y + w_2^2 \sigma_Y^2 .$$





So we have

$$\Phi_Z(\omega) = e^{i\omega(\mu_X w_1 + \mu_Y w_2)} \cdot e^{-\frac{1}{2}\omega^2 (\sigma_X^2 w_1^2 + 2r w_1 w_2 \sigma_X \sigma_Y + \sigma_Y^2 w_2^2)}$$

$$= E\left[ e^{i\omega(w_1 X + w_2 Y)} \right] \quad --- \quad (*)$$

But

$$\Phi_Z(\omega)\Big|_{\omega=1} = E\left[ e^{i(w_1 X + w_2 Y)} \right]$$

$$= e^{i(w_1 \mu_X + w_2 \mu_Y)} e^{-\frac{1}{2}\left[\sigma_X^2 w_1^2 + 2r\, \sigma_X \sigma_Y w_1 w_2 + \sigma_Y^2 w_2^2\right]}$$

**Defn:** The joint moment generating function (mgf) of two j-dist RVs $X$ and $Y$ is

$$\phi_{XY}(s_1, s_2) \triangleq E\left[e^{s_1 X + s_2 Y}\right],$$

where $s_1, s_2 \in \mathbb{R}$ (or $s_1, s_2 \in \mathbb{C}$.)

This is a 2-dimensional bilateral Laplace transform.





**Moment Theorem:** $E[X^j \cdot Y^k]$ can be computed as

$$E[X^j \cdot Y^k] = \frac{\partial^j \partial^k}{\partial s_1^j \, \partial s_2^k} \left( \phi_{XY}(s_1, s_2) \right) \Big|_{\substack{s_1 = 0 \\ s_2 = 0}}$$

$$= \phi_{XY}^{(j,k)}(0,0).$$

**Proof:** Straightforward extension of the one-dimensional case. (exercise.)

## Conditional Distributions

**Defn:** Let $X$ and $Y$ be two j-dist. RVs on $(S, \mathcal{F}, P)$. The joint conditional cdf of $X$ and $Y$ conditioned on $M \in \mathcal{F}$ is

$$F_{XY}(x,y|M) \triangleq P(\{X \le x\} \cap \{Y \le y\} | M)$$

$$= \frac{P(\{X \le x\} \cap \{Y \le y\} \cap M)}{P(M)}.$$

Sometimes, we can describe the event $M$ in terms of $X$ and/or $Y$.



**Example:** Let $M \triangleq \{x_1 < X \le x_2\}$. Find $F_Y(y|M)$.

$$F_Y(y|M) = F_Y(y | \{x_1 < X \le x_2\})$$

$$= P(\{Y \le y\} | \{x_1 < X \le x_2\})$$

$$= \frac{P(\{Y \le y\} \cap \{x_1 < X \le x_2\})}{P(\{x_1 < X \le x_2\})}$$

$$= \frac{F_{XY}(x_2, y) - F_{XY}(x_1, y)}{F_X(x_2) - F_X(x_1)}$$

$$\therefore F_Y(y | \{x_1 < X \le x_2\}) = \frac{F_{XY}(x_2, y) - F_{XY}(x_1, y)}{F_X(x_2) - F_X(x_1)}$$

If we differentiate this w.r.t. $y$, this becomes

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$$f_Y(y \mid \{x_1 < X \le x_2\}) = \frac{\partial}{\partial y} F_Y(y \mid \{x_1 < X \le x_2\})$$

$$= \cdots = \frac{\int_{x_1}^{x_2} f_{XY}(x,y)\,dx}{F_X(x_2) - F_X(x_1)} \quad --- \quad (*)$$

Of particular interest is the case of

$$f_Y(y \mid \{X = x\}) = \lim_{\Delta x \to 0} f_Y(y \mid \{x < X \le x + \Delta x\})$$

So we use $(*)$ and take

$$x_1 = x$$
$$x_2 = x + \Delta x$$

Then

21.13

$$f_Y(y \mid \{x < X \le x + \Delta x\}) \overset{(*)}{=} \frac{\int_x^{x+\Delta x} f_{XY}(\alpha, y)\,d\alpha}{F_X(x+\Delta x) - F_X(x)}$$

and

$$f_Y(y \mid \{X = x\}) = \lim_{\Delta x \to 0} f_Y(y \mid \{x < X \le x + \Delta x\})$$

$$= \lim_{\Delta x \to 0} \frac{\frac{1}{\Delta x} \int_x^{x+\Delta x} f_{XY}(\alpha, y)\,d\alpha}{\frac{1}{\Delta x}\left( F_X(x+\Delta x) - F_X(x) \right)}$$

$$= \lim_{\Delta x \to 0} \frac{\frac{1}{\Delta x}\left( f(x+\Delta x, y) - f(x, y) \right)}{\frac{1}{\Delta x}\left( F_X(x+\Delta x) - F_X(x) \right)}$$

$$= \frac{f_{XY}(x,y)}{f_X(x)}$$

where $\frac{\partial}{\partial x} f(x,y) = f_{XY}(x,y)$

Thus

$$\boxed{f_Y(y \mid \{X=x\}) = \frac{f_{XY}(x,y)}{f_X(x)}}$$

Similarly, by symmetry

$$\boxed{f_X(x \mid \{Y=y\}) = \frac{f_{XY}(x,y)}{f_Y(y)}}$$

We will sometimes use the notation

$$f_X(x \mid \{Y=y\}) = f_X(x \mid y) = f(x \mid y)$$

$$f_Y(y \mid \{X=x\}) = f_Y(y \mid x) = f(y \mid x)$$



n.b. Given that

$$f_X(x \mid \{Y=y\}) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{f_Y(y \mid \{X=x\}) f_X(x)}{f_Y(y)}$$

and

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx$$

$$= \int_{-\infty}^{\infty} f_Y(y \mid \{X=x\}) f_X(x)\,dx,$$

it follows that

$$f_X(x \mid \{Y=y\}) = \frac{f_Y(y \mid \{X=x\})}{\int_{-\infty}^{\infty} f_Y(y \mid \{X=\alpha\}) f_X(\alpha)\,d\alpha}$$

(Bayes' Theorem)

# Iterated Expectation



## Another common situation:

$$E[g(X,Y)] = \iint_{\mathbb{R}^2} g(x,y)\, f_{XY}(x,y)\, dx\, dy$$

$$= \iint_{\mathbb{R}^2} g(x,y)\, f_Y(y|x)\, f_X(x)\, dx\, dy$$

$$= \int_{\mathbb{R}} f_X(x) \underbrace{\left[ \int_{\mathbb{R}} g(x,y)\, f_Y(y|x)\, dy \right]}_{\varphi(x)} dx$$

$$= \int_{\mathbb{R}} f_X(x) \cdot E[g(X,Y) \mid X = x]\, dx = \cdots$$



$$\cdots = \int_{-\infty}^{\infty} f_X(x) \underbrace{E[g(X,Y) | X = x]}_{\varphi(x)} dx$$

$$= \int_{-\infty}^{\infty} f_X(x)\, \varphi(x)\, dx = E_X[\varphi(X)]$$

where $\varphi(x) = E[g(X,Y) \mid \{X = x\}]$

$$\therefore \quad E[g(X,Y)] = E_X[\varphi(X)]$$

$$= E_X[E_Y[g(X,Y) | X]]$$

$$= E[E[g(X,Y) | X]]$$