Session 20

Recall...

20.1

Defn: Two jointly distributed RVs X and Y are jointly Gaussian if their joint pdf (j-pdf) is of the form

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-r^2}} \exp\left\{\frac{-1}{2(1-r^2)}\left[\frac{(x-\mu_X)^2}{\sigma_X^2} - 2r\frac{(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y} + \frac{(y-\mu_Y)^2}{\sigma_Y^2}\right]\right\},$$

where $\mu_X, \mu_Y \in \mathbb{R}$,

$\sigma_X, \sigma_Y > 0$, $\quad$ --- (*)

and

$-1 \leq r \leq 1$. $\quad$ ($-1 < r < 1$ for pdf to exist.)

Recall...

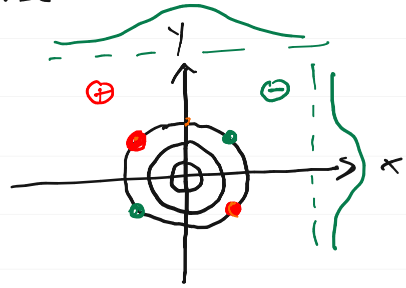
n.b. If X and Y are j-Gaussian, then

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy = \frac{1}{\sqrt{2\pi}\,\sigma_x} \exp\left\{ -\frac{(x-\mu_x)^2}{2\sigma_x^2} \right\}$$

and

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx = \frac{1}{\sqrt{2\pi}\,\sigma_Y} \exp\left\{ -\frac{(y-\mu_Y)^2}{2\sigma_Y^2} \right\}.$$

The converse is not true. (See Papoulis for eg.)



Special case of joint Gaussians

Consider the j-dist Gaussian RVs X and Y having j-pdf

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma^2} \exp\left\{ \frac{-(x^2+y^2)}{2\sigma^2} \right\}.$$

This is a special case of (*) where

$$\mu_x = \mu_Y = 0$$
$$\sigma_x = \sigma_Y = \sigma$$
$$r = 0$$

Suppose I want to find the probability that (X, Y) falls within a distance a of the origin.

Define $D_a = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \le a^2\}$ 20.4

Then

$$P(\{(X,Y)\in D_a\}) = \iint_{D_a} f_{X,Y}(x,y)\,dx\,dy$$

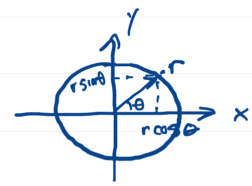
$$= \iint_{D_a} \frac{1}{2\pi\sigma^2} \exp\left\{-\frac{(x^2+y^2)}{2\sigma^2}\right\} dx\,dy$$

(rectangular to polar coordinate change)

$x = r\cos\theta$

$y = r\sin\theta$

$$\underbrace{\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right|}_{\text{Jacobian of the transformation}} = |r| = r$$



$$\Rightarrow P(\{(X,Y)\in D_a\}) = \int_{-\pi}^{\pi}\int_0^a f_{X,Y}(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$ 20.5

$$\int_{-\pi}^{\pi}\int_0^a \frac{\overset{\text{Jacobian}}{r}}{2\pi\sigma^2} \exp\left\{-\frac{(r^2\cos^2\theta + r^2\sin^2\theta)}{2\sigma^2}\right\} dr\,d\theta$$

$\cos^2\theta + \sin^2\theta = 1$

$$= \int_{-\pi}^{\pi}\int_0^a \frac{r}{2\pi\sigma^2} \exp\left\{\frac{-r^2}{2\sigma^2}\right\} dr\,d\theta$$

$$= \int_0^a \underbrace{\frac{r}{\sigma^2} \exp\left\{\frac{-r^2}{2\sigma^2}\right\}}_{\text{This is the pdf of } R = \sqrt{X^2+Y^2}} dr = \underline{1 - e^{-a^2/2\sigma^2}}, \quad a \ge 0.$$

$\leftarrow f_R(r)$

n.b. $f_\Theta(\theta) = \frac{1}{2\pi} \cdot 1_{[-\pi,\pi)}(\theta)$

Statistically Independent RVs

Defn: Two RVs $\mathbb{X}$ and $\mathbb{Y}$ defined on $(\mathcal{S}, \mathcal{F}, P)$ are statistically independent if the events

$\{\mathbb{X} \in A\}$ and $\{\mathbb{Y} \in B\}$ are statistically independent for all $A, B \in \mathcal{B}(\mathbb{R})$.

n.b. Let $A = (-\infty, x]$, $x \in \mathbb{R}$
$B = (-\infty, y]$, $y \in \mathbb{R}$

If $\mathbb{X}$ and $\mathbb{Y}$ are statistically independent, then

$$
\begin{aligned}
F_{\mathbb{X}\mathbb{Y}}(x,y) &= P(\{\mathbb{X} \in A\} \cap \{\mathbb{Y} \in B\}) \\
&= P(\{\mathbb{X} \in A\}) \cdot P(\{\mathbb{Y} \in B\}) \\
&= P(\{\mathbb{X} \in (-\infty, x]\}) \cdot P(\{\mathbb{Y} \in (-\infty, y]\}) \\
&= F_{\mathbb{X}}(x) \cdot F_{\mathbb{Y}}(y)
\end{aligned}
$$

Furthermore

$$
f_{\mathbb{X}\mathbb{Y}}(x,y) = \frac{\partial^2 F_{\mathbb{X}\mathbb{Y}}(x,y)}{\partial x \partial y} = \frac{\partial^2}{\partial x \partial y}\left[F_{\mathbb{X}}(x) \cdot F_{\mathbb{Y}}(y)\right]
$$

$$
= \frac{\partial F_{\mathbb{X}}(x)}{\partial x} \cdot \frac{\partial F_{\mathbb{Y}}(y)}{\partial y} = f_{\mathbb{X}}(x) \cdot f_{\mathbb{Y}}(y)
$$

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Furthermore, if $f_{XY}(x,y) = f_X(x) \cdot f_Y(y)$, then the definition of statistically independent RVs is satisfied.

<u>Proof:</u>

$$P(\{X \in A\} \cap \{Y \in B\}) = \int_B \int_A f_{XY}(x,y)\, dx\, dy$$

For any $A, B \in \mathcal{B}(\mathbb{R})$.

$$= \int_B \int_A f_X(x) \cdot f_Y(y)\, dx\, dy$$

$$= \int_A f_X(x)\, dx \cdot \int_B f_Y(y)\, dy$$

$$= P(\{X \in A\}) \cdot P(\{Y \in B\})$$

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Thus we can take as an equivalent definition ...

<u>Defn'</u>: Two jointly distributed RVs X and Y are <u>statistically independent</u> iff

$$f_{XY}(x,y) = f_X(x) \cdot f_Y(y).$$

Suppose we have

RV X defined on $(S_1, \mathcal{F}_1, P_1)$

and

RV Y defined on $(S_2, \mathcal{F}_2, P_2)$.

We can form a joint experiment $(S, \mathcal{F}, P)$ with

$S = S_1 \times S_2$

$\mathcal{F} = \sigma(\{A \times B : A \in \mathcal{F}_1 \text{ and } B \in \mathcal{F}_2\})$

$P =$ A probability measure consistent with P_1 and P_2.

Then X and Y can be viewed as jointly distributed on $(S, \mathcal{F}, P)$.

<u>Theorem:</u> If random experiments $(S_1, \mathcal{F}_1, P_1)$ and $(S_2, \mathcal{F}_2, P_2)$ are independent experiments, then the j-dist RVs X and Y on $(S, \mathcal{F}, P)$ are statistically independent, where X was defined on $(S_1, \mathcal{F}_1, P_1)$ and Y was defined on $(S_2, \mathcal{F}_2, P_2)$.

One Function of Two RVs: 20.12

Given two j-dist RVs X and Y and a function

$$g(\cdot,\cdot): \mathbb{R}^2 \to \mathbb{R},$$

we can form a new RV

$$Z = g(X, Y).$$

Given $f_{XY}(x,y)$ or $F_{XY}(x,y)$ and $g(x,y)$, we would like to find $f_Z(z)$ or $F_Z(z)$.

Let $D_z \subset \mathbb{R}^2$ $\quad (D_z \in \mathcal{B}(\mathbb{R}^2))$ 20.13

$$D_z \triangleq \{(x,y) \in \mathbb{R}^2 : g(x,y) \leq z\}$$

$$\begin{aligned}\{Z \leq z\} &= \{g(X,Y) \leq z\} \\ &= \{(X,Y) \in D_z\} \\ &= \{\omega \in \mathcal{S} : (X(\omega), Y(\omega)) \in D_z\}\end{aligned}$$

$$\therefore \; F_Z(z) = P(\{Z \le z\}) = P(\{(X,Y) \in D_z\})$$

$$= \iint_{D_z} f_{XY}(x,y)\, dx\, dy$$

$$= \iint_{\mathbb{R}^2} f_{XY}(x,y) \cdot 1_{D_z}((x,y))\, dx\, dy .$$

We can find the p.d.f. of Z as

$$f_Z(z) = \frac{d F_Z(z)}{dz} .$$

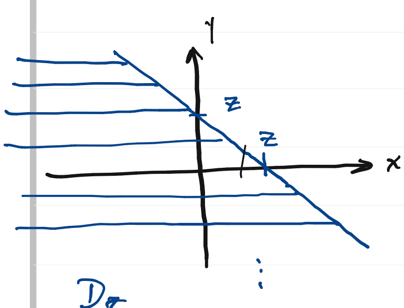
<u>Example:</u> $g(x,y) = x+y$

$$Z = g(X,Y) = X + Y$$

$$F_Z(z) = P(\{Z \le z\}) = P(\{(X,Y) \in D_z\})$$

where $D_z = \{(x,y) \in \mathbb{R}^2 : x+y \le z\}$

$y \le -x + z$



$$F_Z(z) = \iint_{D_z} f_{XY}(x,y)\, dx\, dy$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{z-y} \underbrace{f_{XY}(x,y)}\, dx\, dy$$

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Now if X and Y are statistically independent ($X \perp\!\!\!\perp Y$), this becomes

$$F_Z(z) = \int_{-\infty}^{\infty} \int_{-\infty}^{z-y} f_X(x) \cdot f_Y(y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty} \underbrace{\left[\int_{-\infty}^{z-y} f_X(x)\, dx \right]}_{F_X(z-y)} f_Y(y)\, dy$$

$$= \int_{-\infty}^{\infty} F_X(z-y)\, f_Y(y)\, dy.$$

20.17

Furthermore,

$$f_Z(z) = \frac{dF_Z(z)}{dz} = \frac{d}{dz} \left\{ \int_{-\infty}^{\infty} f_Y(y)\, F_X(z-y)\, dy \right\}$$

$$= \int_{-\infty}^{\infty} f_Y(y) \frac{dF_X(z-y)}{dz}\, dy$$

$$= \int_{-\infty}^{\infty} f_Y(y)\, f_X(z-y)\, dy \quad \text{(convolution integral)}$$

$$(f_Y * f_X)(z) = (f_X * f_Y)(z)$$

Theorem: Let X and Y be two j-dist, independent RVs with marginal pdfs $f_X(x)$ and $f_Y(y)$, respectively. Then the pdf of their sum $Z = X + Y$ is given by the convolution

$$f_Z(z) = (f_X * f_Y)(z)$$

$$= \int_{-\infty}^{\infty} f_X(x)\, f_Y(z-x)\, dx$$

$$= \int_{-\infty}^{\infty} f_Y(y)\, f_X(z-y)\, dy.$$