Session   2

Recall...

There are 3 fundamental set operations we have just defined:

Union: $A \cup B \triangleq \{w \in \Omega : w \in A \text{ or } w \in B\}$

Intersection: $A \cap B \triangleq \{w \in \Omega : w \in A \text{ and } w \in B\}$

Complement: $\bar{A} \triangleq \{w \in \Omega : w \notin A\}$

These are the three fundamental set operations, but there are two other "set difference operations" that are sometimes used:

Defn: An indexed collection of sets is a set of sets

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$$\{ A_i , i \in I \},$$

where I is an index set.

- So $\{ A_i ; i \in I \}$ is a "set of sets" or a "family of sets" or a "collection of sets."

Some Typical index Sets I :

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$N = \{ 1, 2, 3, \ldots \} = $ natural numbers.

$Z_+ = \{ 0, 1, 2, \ldots \} = $ non-negative integers

$Z = \{ \ldots, -2, -1, 0, 1, 2, \ldots \} = $ integers

$I_n = \{ 0, 1, 2, \ldots, n-1 \}$

$R = (-\infty, +\infty) = $ real line

We are interested in the "size" or cardinality of sets:

Defn: A set is finite if it has a finite number of elements. (i.e., its elements can be put in one-to-one correspondence with the numbers $1, 2, \ldots, n$ for some natural number n.

Defn: A set is infinite if it is not finite.

Infinite sets come in two varieties:

Countable and uncountable.

Defn: An infinite set is countable if its elements can be put in one-to-one correspondence with the natural (counting) numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$.

Defn: An infinite set is uncountable if it is not countable.

e.g. The following are examples of uncountable sets:

- $\mathbb{R} = (-\infty, +\infty)$
- $[0,1]$ and $(0,1)$
- $[a,b]$, $[a,b)$, $(a,b]$, (a,b)

$\forall a, b \in \mathbb{R}$ such that $a < b$.

We now expand the definitions of union and intersection beyond simple binary operations:

Defn: Given an indexed family of sets

$$\{A_i ; i \in I\},$$

the __union__ of the sets in the family is

$$\bigcup_{i \in I} A_i \triangleq \{w \in \Omega : w \in A_i \text{ for at least one } i \in I\}$$

the __intersection__ of the sets in the family is

$$\bigcap_{i \in I} A_i \triangleq \{w \in \Omega : w \in A_i \text{ for all } i \in I\}$$

Defn: If $G \subset \Omega$ and $\{A_i ; i \in I\}$ is a family of sets, then if

$$\bigcup_{i \in I} A_i = G$$

we say that $\{A_i ; i \in I\}$ is <u>collectively</u> exhaustive over G.

Defn: A family of sets $\{A_i ; i \in I\}$ is <u>disjoint</u> if

$$A_i \cap A_j = \phi, \quad \forall i, j \in I$$
$$\text{such that } i \neq j.$$

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Defn: A family of sets $\{A_i ; i \in I\}$ is a partition of S if it is disjoint and collectively exhaustive over S.

n.b. $\{A_i ; \in I\}$ is a partition of $G \subset S$ if it is disjoint and

$$\bigcup_{i \in I} A_i = G.$$

Fact: Let $\{A_i ; i \in I\}$ be a partition of S.

Define

$$B_i \overset{\Delta}{=} A_i \cap G \quad, \quad i \in I$$

where $G \subset S$.

Then $\{B_i ; i \in I\}$ is a partition of G.

Proof: Homework

Probability Spaces

A probability space $(\Omega, \mathcal{F}, P)$ is a triple made up of 3 elements:

1. Sample space Ω.

2. A collection of events (subsets of Ω) $\mathcal{F}(\Omega)$

3. The probabilities $P(A)$ for each event in the event space

$$P: \mathcal{F}(\Omega) \longrightarrow [0,1]$$

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The Sample Space Ω

Defn : The sample space Ω is a non-empty set of possible outcomes of a random experiment.

One and only one outcome from the sample space occurs when we perform a random experiment.

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The Event Space $\mathcal{F}(\mathcal{S})$

<u>Defn</u>: The <u>event space</u> $\mathcal{F}(\mathcal{S})$ is a non-empty collection of subsets of $\mathcal{S}$ satisfying the following closure properties.

1. If $A \in \mathcal{F}(\mathcal{S})$, then $\overline{A} \in \mathcal{F}(\mathcal{S})$.

2. For any finite n, if $A_i \in \mathcal{F}(\mathcal{S})$ for $i = 1, 2, \ldots, n$, then
$$\bigcup_{i=1}^{n} A_i \in \mathcal{F}.$$

3. If $A_i \in \mathcal{F}(\mathcal{S})$, $i = 1, 2, 3, \ldots$ then
$$\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}(\mathcal{S}).$$

A sets of subsets satisfying these 3 properties is called a <u>σ-field</u>.

(n.b. If only 1 and 2 hold, you have a <u>field</u> of sets.)

What about intersections?

Proof: Let $A, B \in \mathcal{F}$. Then

$$A \cap B = \overline{(\overline{A \cap B})}$$
$$= \overline{\overline{A} \cup \overline{B}} \in \mathcal{F} \quad (\text{De Morgan})$$

Since $\overline{A} \in \mathcal{F}(\mathcal{S})$ (closure Prop. 1)

$\overline{B} \in \mathcal{F}(\mathcal{S})$ (closure Prop. 1

$\overline{A} \cup \overline{B} \in \mathcal{F}(\mathcal{S})$ (closure Prop. 2)

$\overline{\overline{A} \cup \overline{B}} \in \mathcal{F}(\mathcal{S})$ (closure Prop. 1)

But $\overline{\overline{A} \cup \overline{B}} = A \cup B.$ ∎

It follows from the closure properties that $\phi, \mathcal{S} \in \mathcal{F}(\mathcal{S})$.

Proof: Suppose $A \in \mathcal{F}(\mathcal{S})$ ($\mathcal{F}(\mathcal{S})$ is non-empty)

then $\overline{A} \in \mathcal{F}(\mathcal{S})$ (prop. 1)

Furthermore

$$\mathcal{S} = A \cup \overline{A} \in \mathcal{F}(\mathcal{S}) \quad (\text{prop. 2})$$

$$\overline{\mathcal{S}} = \phi \in \mathcal{F}(\mathcal{S}) \quad (\text{prop. 1})$$

$$\therefore \quad \phi, \mathcal{S} \in \mathcal{F}(\mathcal{S}).$$

Probability Measure

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<u>Defn</u>: A probability measure $P(\cdot)$
(corresponding to $\mathcal{S}$ and $\mathcal{F}(\mathcal{S})$)
is an assingment of a real
number $P(A)$ to each $A \in \mathcal{F}(\mathcal{S})$
satisfying the <u>Axioms of Probability</u>

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Axioms of Probability

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1. $P(A) \geq 0$, $\forall A \in \mathcal{F}(\mathcal{S})$.

2. $P(\mathcal{S}) = 1$.

3. If $A_1, A_2 \in \mathcal{F}(\mathcal{S})$ and $A_1 \cap A_2 = \phi$,
    then $P(A_1 \cup A_2) = P(A_1) + P(A_2)$

- If $\{A_1, \ldots, A_n\}$ (finite) are <u>disjoint</u> $(A_j \cap A_k = \phi)$
    $$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i)$$
    $j \neq k$

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4. If $A_1, A_2, \ldots, A_n, \ldots \in \mathcal{F}(\Omega)$ is a countable collection of disjoint events, then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i).$$

n.b. $P(\cdot)$ is a set function.

$P(\cdot): \mathcal{F}(\Omega) \rightarrow \mathbb{R}.$