

Session 16

Recall,...

Ex. 2 Consider a Poisson RV X
with pmf

16.1

$$P_X(k) = p_k = P\{X=k\} \\ = \frac{e^{-\mu} \mu^k}{k!}, \quad k=0,1,2,\dots; \\ \mu \geq 0.$$

Mean: $E[X] = \sum_{k=0}^{\infty} k \frac{e^{-\mu} \mu^k}{k!} = \sum_{k=1}^{\infty} k \frac{e^{-\mu} \mu^k}{k!}$

$= \mu e^{-\mu} \sum_{k=1}^{\infty} \frac{\mu^{k-1}}{(k-1)!} = \mu e^{-\mu} \sum_{m=0}^{\infty} \frac{\mu^m}{m!}$

$= e^{-\mu} \cdot \mu \cdot e^{\mu} = \boxed{\mu}$

Handwritten notes:
- $k!$ becomes $(k-1)!$
- μ^k becomes μ^m with $m = k-1$
- The sum $\sum_{m=0}^{\infty} \frac{\mu^m}{m!}$ is identified as e^{μ}

Variance: $\text{var}(X) = E[X^2] - (E[X])^2$
 $= E[X^2] - \mu^2$

16.2

So $E[X^2] = \sum_{k=0}^{\infty} k^2 \frac{e^{-\mu} \mu^k}{k!} = \sum_{k=1}^{\infty} k \frac{e^{-\mu} \mu^k}{(k-1)!}$
 $= \sum_{k=1}^{\infty} \frac{k e^{-\mu} \mu^k}{(k-1)!} = \sum_{s=0}^{\infty} \frac{(s+1) e^{-\mu} \mu^{s+1}}{s!}$
 $= \mu \sum_{s=0}^{\infty} \frac{s e^{-\mu} \mu^s}{s!} + \mu \sum_{s=0}^{\infty} \frac{e^{-\mu} \mu^s}{s!}$
 $= \mu \cdot \mu + \mu \cdot 1 = \mu^2 + \mu$

$\therefore \text{var}(X) = \mu^2 + \mu - \mu^2 = \mu$

$\text{Std Dev}(X) = \sqrt{\text{var}(X)} = \sqrt{\mu}$

The Characteristic Function

16.3

Defn: Let X be a R.V. on $(\mathcal{S}, \mathcal{F}, P)$.

The characteristic function of X

is

$$\Phi_X(\omega) \triangleq E[e^{i\omega X}], \omega \in \mathbb{R}$$

If $f_X(x)$ is the pdf of X , then

$$\Phi_X(\omega) = \int_{-\infty}^{\infty} e^{i\omega x} f_X(x) dx.$$

$$\Phi_X(\omega) : \mathbb{R} \rightarrow \mathbb{C}.$$

$$\Phi_*(\omega) = \int_{-\infty}^{\infty} f_*(x) e^{+i\omega x} dx$$

16.4

is a Fourier transform of the pdf $f_*(x)$

n.b., $i = \sqrt{-1}$. Engineers often use "j" instead of "i", but we will use "i".

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$$\Phi_*(\omega) = \int_{-\infty}^{\infty} f_*(x) e^{+i\omega x} dx$$

16.5

n.b. $e^{i\omega x} = \cos \omega x + i \sin \omega x$ (Euler's formula)

$$\left(\begin{array}{l} e^{i\theta} = \cos \theta + i \sin \theta \\ \text{set } \theta = \pi \Rightarrow e^{i\pi} = -1 \\ \Rightarrow e^{i\pi} + 1 = 0 \end{array} \right)$$

Paul J. Nahin, Dr. Euler's Fabulous Formula,
Princeton U. Press, 2006.

Aside

Also note from Euler's formula

$$\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad \text{and} \quad \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

From which it is easy to derive many trig formulas

$$\cos \alpha \cos \beta = \frac{1}{2} [\cos(\alpha - \beta) + \cos(\alpha + \beta)]$$

$$\sin \alpha \sin \beta = \dots, \quad \cos \alpha \cdot \sin \beta = \dots$$

Anyway, we have

16.6

$$\begin{aligned} E[e^{i\omega X}] &= E[\cos \omega X + i \sin \omega X] \\ &= E[\cos \omega X] + i E[\sin \omega X] \end{aligned}$$

$$\begin{aligned} \text{--- n.b.} \quad |\Phi_X(\omega)| &= \left| \int_{-\infty}^{\infty} f_X(x) e^{i\omega x} dx \right| \leq \int_{-\infty}^{\infty} |f_X(x) e^{i\omega x}| dx \\ &= \int_{-\infty}^{\infty} \underbrace{|f_X(x)|}_{f_X(x)} \cdot \underbrace{|e^{i\omega x}|}_{1} dx = \int_{-\infty}^{\infty} f_X(x) dx = 1. \end{aligned}$$

$$\Rightarrow |\Phi_X(\omega)| \leq \Phi_X(0) = 1.$$

$\therefore \Phi_X(\omega)$ is well defined for any $f_X(x)$

There is a corresponding inverse Fourier transform relationship:

16.7

$$f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_X(\omega) e^{-i\omega x} d\omega$$

$\therefore$ Given $\Phi_X(\omega)$, we can find $f_X(x)$, which is a complete probabilistic description of X

$\Rightarrow \Phi_X(\omega)$ is a complete probabilistic description of X .

Fact: Suppose $\Phi_0(\omega)$ is known to be the characteristic function of RV X and RV Y :

16.8

$$\Phi_X(\omega) = \Phi_Y(\omega) \equiv \Phi_0(\omega).$$

Then X and Y have identical pdfs and cdfs:

$$f_X(x) = f_Y(x), \quad \forall x \in \mathbb{R}$$

$$F_X(x) = F_Y(x), \quad \forall x \in \mathbb{R}.$$

This does not mean $X(\cdot) = Y(\cdot)$.

n.b.: Just because two RVs have the same pdf does not mean that they are equal:

16.9

Let RV X have pdf

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{x^2}{2\sigma^2}\right\}$$

Define $Y = -X$. It can be shown

$$\text{that } f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{y^2}{2\sigma^2}\right\}$$

(i.e., $f_X(x) = f_Y(x)$, $\forall x \in \mathbb{R}$, and yet the only time $X(\omega) = Y(\omega)$ is when $X(\omega) = 0$).

$$P(\{X=Y\}) = P(\{X=0\}) = 0$$

Example: If X is an exponentially distributed RV with mean μ , find $\Phi_X(\omega)$. 16.10

$$\begin{aligned}\Phi_X(\omega) &= E[e^{i\omega X}] = \int_{-\infty}^{\infty} e^{i\omega x} f_X(x) dx \\&= \int_0^{\infty} e^{i\omega x} \cdot \frac{1}{\mu} \exp\left(-\frac{x}{\mu}\right) \cdot 1_{(0, \infty)}(x) dx \\&= \int_0^{\infty} e^{i\omega x} \cdot \frac{1}{\mu} e^{-x/\mu} dx = \frac{1}{\mu} \int_0^{\infty} e^{(i\omega - \frac{1}{\mu})x} dx \\&= \dots = \frac{\frac{1}{\mu}}{\frac{1}{\mu} - i\omega} = \frac{1}{1 - i\omega\mu} = (1 - i\omega\mu)^{-1}\end{aligned}$$

Useful Property: If X is a RV with characteristic function $\Phi_X(\omega)$, and 16.11

$$Y = aX + b, \quad a, b \in \mathbb{R},$$

then $\Phi_Y(\omega) = e^{i\omega b} \Phi_X(a\omega).$

Proof:
$$\begin{aligned}\Phi_Y(\omega) &= E[e^{i\omega Y}] = E[e^{i\omega(aX+b)}] \\&= E[e^{i\omega a X} \cdot e^{i\omega b}] = e^{i\omega b} E[e^{i(\omega a)X}] \\&= e^{i\omega b} \cdot \Phi_X(a\omega).\end{aligned}$$

Defn: The moment generating function of a RV X is defined as

$$\phi_X(s) \triangleq E[e^{sX}],$$

where $s \in \mathbb{R}$ (or $s \in \mathbb{C}$.)

- In elementary courses, s is taken to be real
- If we take $s \in \mathbb{C}$, $\phi_X(s)$ is a bilateral Laplace transform.

$$\phi_X(s) \triangleq E[e^{sX}] = \int_{-\infty}^{\infty} f_X(x) e^{sx} dx$$

If we let $s \in \mathbb{C}$:

- $\overline{\phi_X(\omega)} = \phi_X(\underline{i\omega})$
- $\phi_X(\cdot) : \mathbb{C} \rightarrow \mathbb{C}$

If $s \in \mathbb{R}$: then

$$\phi_X(\cdot) : \mathbb{R} \rightarrow \mathbb{R}$$

Suppose I want to compute

16.14

$E[X^n]$ ~ "The n -th moment of X "

The following theorem is useful:

Moment Theorem: Given a RV X with mgf $\phi_X(s)$, the n -th moment of X is given by

$$E[X^n] = \left. \frac{d^n \phi_X(s)}{ds^n} \right|_{s=0} = \phi_X^{(n)}(0).$$

(Alternatively, we could compute $E[X^n] = \int_{-\infty}^{\infty} x^n f_X(x) dx$.)

Proof: Consider $\phi_X(s) = E[e^{sX}]$.

16.15

$$\begin{aligned} \phi_X^{(n)}(s) &= \frac{d^n}{ds^n} (E[e^{sX}]) = E\left[\frac{d^n e^{sX}}{ds^n}\right] \\ &= E[X^n e^{sX}] \end{aligned}$$

So setting $s=0$, we have

$$\phi_X^{(n)}(0) = E\left[X^n \underbrace{e^{0 \cdot X}}_1\right] = E[X^n].$$

Example: Suppose X is an exponentially distributed RV with mean μ . Find its variance.

16.16

$$\phi_X(\omega) = \frac{1}{1 - i\omega\mu} = (1 - i\omega\mu)^{-1}$$

$$\phi_X(s) = \frac{1}{1 - s\mu} = (1 - s\mu)^{-1}$$

$$\text{var}(X^2) = E[X^2] - (E[X])^2 = E[X^2] - \mu^2$$

$$E[X^2] \stackrel{\text{m.t.}}{=} \left. \frac{d^2 \phi_X(s)}{ds^2} \right|_{s=0}$$

$$= \left. \frac{d^2}{ds^2} [(1 - s\mu)^{-1}] \right|_{s=0}$$

$$\dots = \left. \frac{d}{ds} [-(1 - s\mu)^{-2} (-\mu)] \right|_{s=0}$$

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$$= \left. [-2\mu (1 - s\mu)^{-3} (-\mu)] \right|_{s=0}$$

$$= -2\mu \cdot 1 \cdot (-\mu) = 2\mu^2$$

$$\text{var}(X) = 2\mu^2 - (\mu)^2 = \boxed{\mu^2}$$

We can use the characteristic function in the moment theorem:

16.18

$$E[X^n] = \frac{1}{i^n} \left. \frac{d^n \Phi_X(\omega)}{d\omega^n} \right|_{\omega=0}$$

or (identifying $\Phi_X(\omega)$ with $\phi_X(i\omega)$)

$$E[X^n] = \left. \frac{d^n \Phi_X(\omega)}{d(i\omega)^n} \right|_{i\omega=0}$$

e.g.

For the exponential case, the char. fcn. is

16.19

$$\Phi_X(\omega) = (1 - i\omega\mu)^{-1}$$

$$E[X] = \frac{d}{d(i\omega)} [(1 - i\omega\mu)^{-1}]$$

$$= -(1 - i\omega\mu)^{-2} (-\mu) \Big|_{i\omega=0}$$

$$= \frac{\mu}{(1 - i\omega\mu)^2} \Big|_{i\omega=0} = \frac{\mu}{1} = \boxed{\mu}$$