Session 16

Recall...

Mean, Variance and Expectation

16.1

Defn: The mean or expected value of a RV X with pdf $f_X(x)$ is

$$E[X] \triangleq \int_{-\infty}^{\infty} x f_X(x)\, dx.$$

Recall...

16.2

$\therefore$ For a discrete RV X, we have

$$\boxed{E[X] = \sum_k x_k\, p_X(x_k).}$$

If you know about Riemann-Stiltjes integrals, you can write

Don't worry about this (Riemann-Stiltjes)

$$E[X] = \int_{-\infty}^{\infty} x\, dF_X(x)$$

$$= \begin{cases} \sum_k x_k\, p_X(x_k) & (\text{discrete RV } X) \\ \int_{-\infty}^{\infty} x\, f_X(x)\, dx & (\text{continuous RV } X) \end{cases}$$

Recall...

16.3

Defn: Let X be a RV on $(\mathcal{S}, \mathcal{F}, P)$ and let $M \in \mathcal{F}$. Then the conditional mean of X conditioned on M is

$$E[X|M] \triangleq \int_{-\infty}^{\infty} x\, f_X(x|M)\, dx.$$

(n.b. If X is discrete, we have the conditional pmf $p_X(x_k|M) = P(\{X = x_k\}|M)$, and then

$$E[X|M] = \int_{-\infty}^{\infty} x f_X(x|M)\, dx = \int_{-\infty}^{\infty} x \left(\sum_k p_X(x_k|M)\, \delta(x - x_k)\right) dx$$

$$= \sum_k p_X(x_k|M) \cdot \int_{-\infty}^{\infty} x\, \delta(x - x_k)\, dx = \sum_k x_k\, p_X(x_k|M) \Big)$$

Recall...

Example: Let X be an exponentially distributed RV with pdf

16.4

$$f_X(x) = \frac{1}{\mu} \exp\left\{-\frac{x}{\mu}\right\} \mathbb{1}_{[0,\infty)}(x), \quad \mu > 0$$

What is $E[X]$?

$$E[X] = \int_{-\infty}^{\infty} x\, f_X(x)\, dx = \int_0^{\infty} x \cdot \frac{1}{\mu} e^{-x/\mu}\, dx$$

$$\underset{\text{(exercise)}}{\overset{\text{int. by parts}}{=}} \cdots = \left[-x e^{-x/\mu} - \mu e^{-x/\mu}\right]_0^{\infty} = \boxed{\mu}$$

Now let's consider the conditional mean

$$E[X \mid M], \quad \text{where } M = \{X > \mu\}.$$

$$E[X \mid \{X > \mu\}] = \int_{-\infty}^{\infty} x\, f_X(x \mid \{X > \mu\})\, dx$$

16.5

It is straightforward to show that

$$f_X(x \mid \{X > \mu\}) = \frac{\frac{1}{\mu} \exp\left(\frac{-x}{\mu}\right) \cdot \mathbb{1}_{(\mu,\infty)}(x)}{e^{-\mu/\mu}}$$

(Exercise)

$$= \frac{1}{\mu} \exp\left\{-\frac{(x-\mu)}{\mu}\right\} \cdot \mathbb{1}_{(\mu,\infty)}(x)$$

$$\therefore\ E[X \mid \{X > \mu\}] = \int_{-\infty}^{\infty} x\, f_X(x \mid \{X > \mu\})\, dx$$

$$= \int_{\mu}^{\infty} x \cdot \frac{1}{\mu} \exp\left\{\frac{-(x-\mu)}{\mu}\right\} dx$$

let $\left.\begin{aligned} r &= x - \mu \\ x &= r + \mu \\ dx &= dr \end{aligned}\right\} \Rightarrow = \int_0^{\infty} (r+\mu) \cdot \frac{1}{\mu} \exp\left\{\frac{-r}{\mu}\right\} dr = \cdots$

16.6

$$= \underbrace{\int_0^\infty \frac{r}{\mu} \exp\left(\frac{-r}{\mu}\right) dr}_{\mu} + \underbrace{\mu \int_0^\infty \frac{1}{\mu} \exp\left(\frac{-r}{\mu}\right) dr}_{\mu \cdot 1}$$

$$= \boxed{2\mu}$$

n.b. $E[X] \neq E[X \mid \{X > \mu\}]$

More generally

$$E[X \mid M] \neq E[X].$$

16.7

Suppose we have a RV X defined on $(\mathcal{S}, \mathcal{F}, P)$ and having pdf $f_X(x)$

Now suppose I have a new RV

$$Y = g(X)$$

where $g : \mathbb{R} \to \mathbb{R}$. What is $E[Y]$?

It appears we must first find $f_Y(y)$, and then compute

$$E[Y] = \int_{-\infty}^{\infty} y f_Y(y)\, dy.$$

This will work, but there is an easier way.

16.8

Fact: Let X be a RV and define the new RV $Y = g(X)$.

Then

$$E[Y] = E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx.$$

n.b. $\int_{-\infty}^{\infty} g(x) f_X(x)\, dx = \int_{-\infty}^{\infty} y\, f_Y(y)\, dy$

Proof: Outlined in Papoulis.

16.9

Basic idea: Assume $g(x)$ is monotonically increasing

$y = g(x)$, $\frac{dx}{dy} = \left|\frac{dx}{dy}\right| > 0$

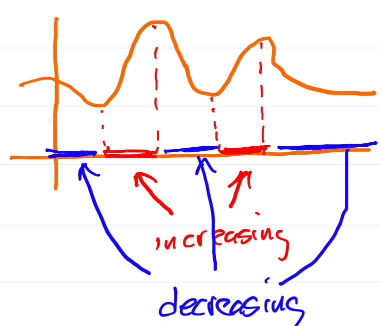
So $E[Y] \triangleq \int_{-\infty}^{\infty} y\, f_Y(y)\, dy$

$$= \int_{-\infty}^{\infty} \underbrace{g(x)}_{y} \cdot \underbrace{f_X(x) \frac{dx}{dy}}_{f_Y(y)} \cancel{dy}$$

$$= \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$$

Similar for monotonically decreasing function

(You are not responsible for proof.)



A general function can be broken into monotonically increasing and decreasing segments (and flat segments)

"Defn:" The expected value of a function $g(X)$ of a RV X is

$$E[g(X)] \triangleq \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$$

n.b. If X is a discrete RV, this becomes

$$E[g(X)] = \sum_k g(x_k)\, p_X(x_k)$$

Linearity of Expectation

Let $g_1(X)$ and $g_2(X)$ be two functions of a RV X and let α and β be two constants ($\alpha, \beta \in \mathbb{R}$ or $\mathbb{C}$).

Then

$$E[\alpha g_1(X) + \beta g_2(X)]$$

$$= \alpha E[g_1(X)] + \beta E[g_2(X)]$$

Proof: Exercise

<u>Defn:</u> The <u>variance</u> of a RV X is defined as

$$\mathrm{var}(X) \triangleq E[(X-\bar{X})^2]$$

$$= \int_{-\infty}^{\infty} (x-\bar{X})^2 f_X(x)\, dx ,$$

where $\bar{X} = E[X]$

<u>Defn</u>: The positive square root of the variance of X is called the <u>standard deviation</u> of X:

$$\mathrm{StDev}(X) = \sigma_X = \sqrt{\mathrm{var}(X)}$$

<u>n.b.</u> $\mathrm{var}(X) \triangleq E[(X-\bar{X})^2]$

$$= E[(X^2 - 2\bar{X}X + (\bar{X})^2]$$

$$= E[X^2] - 2\bar{X}E[X] + (\bar{X})^2$$

$$= E[X^2] - 2\bar{X}\cdot\bar{X} + (\bar{X})^2$$

$$= E[X^2] - 2(\bar{X})^2 + (\bar{X})^2$$

$$= E[X^2] - (E[X])^2$$

$$\therefore \boxed{\mathrm{Var}(X) = E[X^2] - (E[X])^2}$$

Ex. 1: Consider a Gaussian RV $\mathbb{X}$ with pdf

$$f_{\mathbb{X}}(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left\{ -\frac{(x-\mu)^2}{2\sigma^2} \right\},$$

where $\mu \in \mathbb{R}$ and $\sigma > 0$.

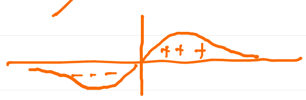
Mean:

$$E[\mathbb{X}] = \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi}\,\sigma} \exp\left\{ -\frac{(x-\mu)^2}{2\sigma^2} \right\} dx$$

Let $r = x - \mu \Rightarrow x = r + \mu \Rightarrow dx = dr$

$$= \int_{-\infty}^{\infty} \frac{r+\mu}{\sqrt{2\pi}\,\sigma} \exp\left\{ \frac{-r^2}{2\sigma^2} \right\} dr$$

$$= \int_{-\infty}^{\infty} \frac{r}{\sqrt{2\pi}\,\sigma} \exp\left\{ \frac{-r^2}{2\sigma^2} \right\} dr + \mu \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left\{ \frac{-r^2}{2\sigma^2} \right\} dr}$$



$$= 0 + \mu \cdot 1 = \boxed{\mu}$$

variance: $\mathrm{var}(\mathbb{X}) = E[\mathbb{X}^2] - \mu^2$

$$E[\mathbb{X}^2] = \int_{-\infty}^{\infty} x^2 \cdot \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left\{ -\frac{(x-\mu)^2}{2\sigma^2} \right\} dx$$

let $r = x - \mu$
$x = r + \mu$
$dx = dr$

$$= \int_{-\infty}^{\infty} (r+\mu)^2 \cdot \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left\{ \frac{-r^2}{2\sigma^2} \right\} dr$$

(exercise)

$$= \cdots = \sigma^2 + \mu^2$$

$$\Rightarrow \mathrm{var}(\mathbb{X}) = \sigma^2 + \mu^2 - (\mu)^2 = \boxed{\sigma^2}$$

$\therefore$ A Gaussian RV with pdf

$$f_{\mathbb{X}}(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left\{ -\frac{(x-\mu)^2}{2\sigma^2} \right\}$$

has mean μ and variance σ^2

Ex. 2 Consider the Poisson RV X with pmf

$$P_X(k) = p_k = P(\{X = k\})$$
$$= \frac{e^{-\mu}\mu^k}{k!}, \quad k = 0, 1, 2, \ldots, \quad \mu > 0$$

Compute the mean an variance of X.

mean:
$$E[X] = \sum_{\substack{\cancel{k=0}\\k=1}}^{\infty} \cancel{k}\frac{e^{-\mu}\mu^k}{\cancel{k!}_{(k-1)!}} = \sum_{k=1}^{\infty} \frac{e^{-\mu}\mu^k}{(k-1)!}$$
$$= \mu e^{-\mu} \sum_{k=1}^{\infty} \frac{\mu^{k-1}}{(k-1)!} = e^{-\mu}\mu \underbrace{\sum_{m=0}^{\infty} \frac{\mu^m}{m!}}_{e^{\mu}}$$
$$= e^{-\mu}\cdot\mu\cdot e^{\mu} = \boxed{\mu}$$

variance:
$$\text{var}(X) = E[X^2] - (E[X])^2$$
$$= E[X^2] - \mu^2$$

So
$$E[X^2] = \sum_{\substack{\cancel{k=0}\\k=1}}^{\infty} k^2\frac{e^{-\mu}\mu^k}{k!} = \sum_{k=1}^{\infty} \frac{k e^{-\mu}\mu^k}{(k-1)!}$$

$s = k-1 \Rightarrow k = s+1$

$$= \sum_{s=0}^{\infty} \frac{(s+1)e^{-\mu}\mu^{s+1}}{s!}$$
$$= \mu \underbrace{\sum_{s=0}^{\infty} \frac{s e^{-\mu}\mu^s}{s!}}_{\mu} + \mu \underbrace{\sum_{k=0}^{\infty} \frac{e^{-\mu}\mu^s}{s!}}$$
$$= \mu\cdot\mu + \mu\cdot 1 = \mu^2 + \mu$$

$$\therefore \text{var}(X) = \mu^2 + \mu - \mu^2 = \boxed{\mu}$$

The Characteristic Function

Defn: Let X be a R.V. on $(S, \mathcal{F}, P)$.
The characteristic function of X is

$$\Phi_X(\omega) \triangleq E\left[e^{i\omega X}\right], \quad \omega \in \mathbb{R}.$$

If $f_X(x)$ is the pdf of X, then

$$\Phi_X(\omega) = \int_{-\infty}^{\infty} e^{i\omega x} f_X(x)\, dx.$$

$$\Phi_X(\omega) : \mathbb{R} \to \mathbb{C}.$$

$$\Phi_X(\omega) = \int_{-\infty}^{\infty} f_X(x)\, e^{+i\omega x}\, dx$$

is a Fourier transform of the pdf $f_X(x)$

n.b., $i = \sqrt{-1}$. Engineers often use "j" instead of "i", but we will use "i".