

Session 14

Recall...

The Total Prob. Law and Bayes Theorem

14.1

Given a RV X on (Ω, \mathcal{F}, P) , let $\{A_1, \dots, A_n\}$ be a partition of Ω , with $A_k \in \mathcal{F}$, $k=1, \dots, n$.

Then

$$P(\{X \leq x\}) = P(\{X \leq x\} | A_1) P(A_1) + P(\{X \leq x\} | A_2) P(A_2) + \dots + P(\{X \leq x\} | A_n) P(A_n) \quad \dots (1)$$

But note that

$$P(\{X \leq x\}) = F_X(x)$$

$$P(\{X \leq x\} | A_k) = F_X(x | A_k)$$

Recall...

$$\therefore F_{\#}(x) = F_{\#}(x|A_1)P(A_1) + F_{\#}(x|A_2)P(A_2) + \dots + F_{\#}(x|A_n)P(A_n) \quad \text{--- (1A)}$$

note $f_{\#}(x) = \frac{dF_{\#}(x)}{dx}$
and $f_{\#}(x|A_k) = \frac{dF_{\#}(x|A_k)}{dx}$

$$\Rightarrow f_{\#}(x) = f_{\#}(x|A_1)P(A_1) + f_{\#}(x|A_2)P(A_2) + \dots + f_{\#}(x|A_n)P(A_n) \quad \text{--- (1B)}$$

13.11

Recall...

2. Recall Bayes Formula:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

Let $B = \{x \leq x\}$. Then

$$\begin{aligned} P(A|\{x \leq x\}) &= \frac{P(\{x \leq x\}|A)P(A)}{P(\{x \leq x\})} \\ &= \frac{F_{\#}(x|A)P(A)}{F_{\#}(x)} \end{aligned}$$

$$\therefore P(A|\{x \leq x\}) = \frac{F_{\#}(x|A)P(A)}{F_{\#}(x)}$$

--- (2)

14.3

Now consider $P(A|B) = \frac{P(B|A)P(A)}{P(B)}$

14.4

Let $B = \{x_1 < X \leq x_2\}$, $x_1 < x_2$

Then we have

$$\begin{aligned} P(A|\{x_1 < X \leq x_2\}) &= \frac{P(\{x_1 < X \leq x_2\}|A)P(A)}{P(\{x_1 < X \leq x_2\})} \\ &= \frac{(F_X(x_2|A) - F_X(x_1|A))P(A)}{F_X(x_2) - F_X(x_1)} \quad \text{--- (3)} \end{aligned}$$

What about $P(A|\{X=x\})$

14.5

Suppose $B = \{X=x\}$. What is $P(A|B)$?

$$P(A|\{X=x\}) \stackrel{?}{=} \frac{P(A \cap \{X=x\})}{P(\{X=x\})}$$

n.b. $P(\{X=x\}) = 0$ if RV X is continuous

and $A \cap \{X=x\} \subset \{X=x\}$

$$\begin{aligned} &= \underbrace{P(A \cap \{X=x\})}_{\substack{\parallel \\ 0}} \leq P(\{X=x\}) = 0 \\ &\quad \text{if } X \text{ is abs. continuous} \end{aligned}$$

So we have

14.6

$$P(A | \{X=x\}) = \frac{P(A \cap \{X=x\})}{P(\{X=x\})} = \frac{0}{0} \text{ - undefined.}$$

But clearly the probability $P(A | \{X=x\})$ is meaningful.

Approach:

14.7

Consider $P(A | \{x_1 < X \leq x_2\})$

where

$$x_1 = x$$
$$x_2 = x + \Delta x$$

Then we consider

$$\lim_{\Delta x \rightarrow 0} P(A | \{x < X \leq x + \Delta x\}) = P(A | \{X=x\})$$

So

14.8

$$P(A | \{X=x\}) = \lim_{\Delta x \rightarrow 0} P(A | \{x < X \leq x + \Delta x\})$$

$$\stackrel{(3)}{=} \lim_{\Delta x \rightarrow 0} \left[\frac{F_{*}(x + \Delta x | A) - F_{*}(x | A)}{F_{\#}(x + \Delta x) - F_{\#}(x)} \right] P(A)$$

$$= \lim_{\Delta x \rightarrow 0} \left[\frac{\frac{F_{\#}(x + \Delta x | A) - F_{\#}(x | A)}{\Delta x}}{\frac{F_{\#}(x + \Delta x) - F_{\#}(x)}{\Delta x}} \right] P(A)$$

Aside:

14.9

$$\text{Fact: } \lim_{p \rightarrow 0} \frac{A(p)}{B(p)} = \frac{(\lim_{p \rightarrow 0} A(p))}{(\lim_{p \rightarrow 0} B(p))}$$

Assuming $\lim_{p \rightarrow 0} A(p)$, $\lim_{p \rightarrow 0} B(p)$ and

$\lim_{p \rightarrow 0} \frac{A(p)}{B(p)}$ are well defined

$$= \dots = \frac{F_{*}(x | A)}{f_{\#}(x)} P(A),$$

14.10

$$\therefore P(A | \{X=x\}) = \frac{f_{X|A}(x|A)}{f_X(x)} P(A)$$

[4]

From (4), multiplying both sides by $f_X(x)$, we get

$$P(A | \{X=x\}) f_X(x) = f_{X|A}(x|A) P(A)$$

$$\Rightarrow \int_{-\infty}^{\infty} P(A | \{X=x\}) f_X(x) dx = P(A) \underbrace{\int_{-\infty}^{\infty} f_{X|A}(x|A) dx}_{1}$$

$$\Rightarrow P(A) = \int_{-\infty}^{\infty} P(A | \{X=x\}) f_X(x) dx.$$

14.11

$$\therefore P(A) = \int_{-\infty}^{\infty} P(A | \{X=x\}) f_X(x) dx$$

[5]

Total Probability Law

Furthermore from [4]

14.12

$$f_{\#}(x|A) = \frac{P(A|\{X=x\}) f_{\#}(x)}{P(A)}$$

$$\therefore f_{\#}(x|A) = \frac{P(A|\{X=x\}) f_{\#}(x)}{\int_{-\infty}^{\infty} P(A|\{X=\alpha\}) f_{\#}(\alpha) d\alpha}$$

Bayes Theorem

--- [6]

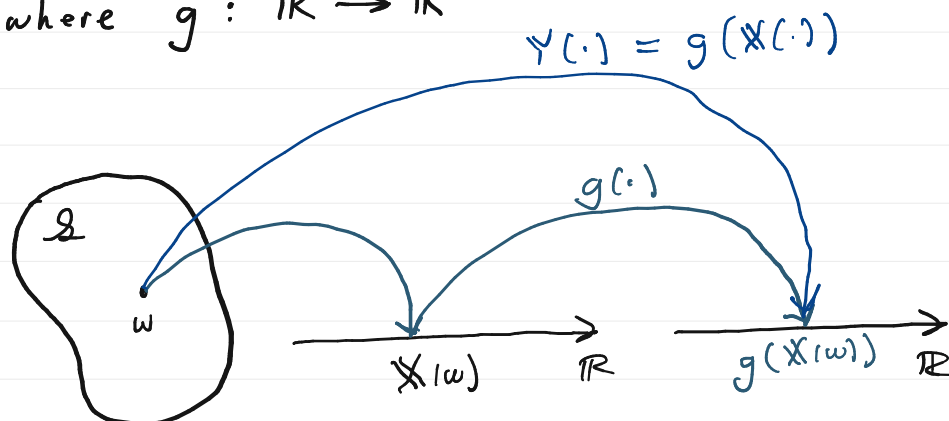
Functions of a Random Variable

14.13

Assume X is a RV on $(\mathcal{S}, \mathcal{F}, P)$. Now assume we have a function

$$Y = g(X),$$

where $g: \mathbb{R} \rightarrow \mathbb{R}$



So $Y(\cdot) = g(X(\cdot)) : \mathcal{S} \rightarrow \mathbb{R}$.

14.14

It appears to be a random variable.

Is it?

Recall: From the defn. of a RV, $Y : \mathcal{S} \rightarrow \mathbb{R}$ is a RV if

$$Y^{-1}(A) = \{\omega \in \mathcal{S} : Y(\omega) \in A\} \in \mathcal{F}, \forall A \in \mathcal{B}(\mathbb{R}),$$

where \mathcal{F} is the event space of $(\mathcal{S}, \mathcal{F}, P)$.

For $Y = g(X)$ to be measurable

14.15

(i.e., an RV) $g(\cdot)$ must satisfy the following properties:

1. The domain of $g(\cdot)$ must contain the range space of X .

2. For each $y \in \mathbb{R}$, the set $R_y \triangleq \{x \in \mathbb{R} : g(x) \leq y\}$ must be a Borel set.

3. The events $\{g(X) = \pm\infty\}$ must have probability 0.

Any function $g(\cdot)$ satisfying these 3 properties is called a Baire function 14.16

For such functions $g(\cdot)$,

$$Y = g(X)$$

is a valid random variable.

All functions we typically encounter in engineering are Baire functions.

The Distribution of $Y = g(X)$

14.17

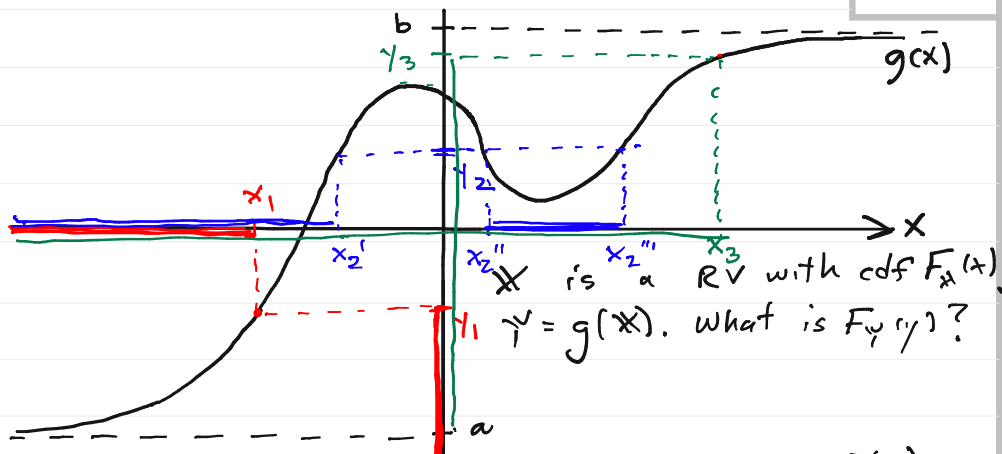
Given a RV X on $(\mathcal{S}, \mathcal{F}, P)$ with cdf $F_X(x)$, and given a function $g: \mathbb{R} \rightarrow \mathbb{R}$, we define $Y = g(X)$.

$$\begin{aligned} \text{Find } F_Y(y) &= P(\{Y \leq y\}) = P(\{g(X) \leq y\}) \\ &= P(\{X \in g^{-1}(-\infty, y]\}) \\ &= P_X(g^{-1}(-\infty, y]) \end{aligned}$$

This tells us how to find $F_Y(y)$ for any $y \in \mathbb{R}$.

1. Consider a generic function $g(\cdot)$:

14.18



Consider y_1 : $F_Y(y_1) = P(\{Y \leq y_1\}) = P(\{X \leq x_1\}) = F_X(x_1)$

Consider y_2 : $F_Y(y_2) = P(\{Y \leq y_2\}) = P(\{X \leq x_2'\} \cup \{x_2'' \leq X \leq x_2'''\})$
 ↑ disjoint ↑

14.19

$$\begin{aligned} \therefore F_Y(y_2) &= P(\{X \leq x_2'\}) + P(\{x_2'' \leq X \leq x_2'''\}) \\ &= F_X(x_2') + (F_X(x_2''') - F_X(x_2'')) \end{aligned}$$

If X is not absolutely continuous, we have to think carefully about the situation at $x = x_2''$.

So to determine $F_Y(y)$ completely, we must do this for all $y \in \mathbb{R}$.

This can be easy or difficult depending the complexity of $g(\cdot)$

Ex. A $Y = aX + b$, $a, b \in \mathbb{R}$

14.20

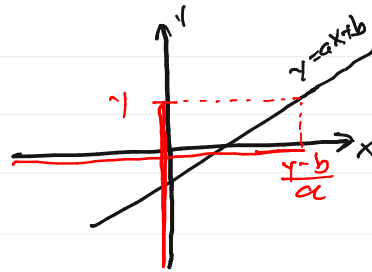
$\Rightarrow g(x) = ax + b$ (affine transformation)

two cases: $a \geq 0$.

(i) $a > 0$

$$\begin{aligned} F_Y(y) &= P(\{X \leq y\}) = P(\{aX + b \leq y\}) \\ &= P(\{X \leq \frac{y-b}{a}\}) \\ &= F_X\left(\frac{y-b}{a}\right) \end{aligned}$$

$$\begin{aligned} f_Y(y) &= \frac{dF_Y(y)}{dy} = f_X\left(\frac{y-b}{a}\right) \cdot \frac{1}{a} \\ &= \frac{1}{a} f_X\left(\frac{y-b}{a}\right) \end{aligned}$$



(ii) $a < 0$: $F_Y(y) = P(\{X \leq y\})$

14.21

$$\begin{aligned} &= P(\{aX + b \leq y\}) \\ &= P(\{X \geq \frac{y-b}{a}\}) \\ &= P(\{X \geq \frac{y-b}{a}\}) \\ &= 1 - F_X\left(\frac{y-b}{a}\right) \end{aligned}$$

