Session 13

Recall...

13.1

Defn: The conditional cdf of the RV $\mathbb{X}$ conditioned on $M \in \mathcal{F}$ is

$$F_{\mathbb{X}}(x \mid M) = P(\{\mathbb{X} \le x\} \mid M)$$

$$= \frac{P(\{\mathbb{X} \le x\} \cap M)}{P(M)}, \quad x \in \mathbb{R}$$

The definition of $F_{\mathbb{X}}(x \mid M)$ is just like the definition of $F_{\mathbb{X}}(x)$, except we use the conditional probability measure $P(\cdot \mid M)$ instead of $P(\cdot)$.

$$(\mathcal{S}, \mathcal{F}, P) \xrightarrow{M \text{ occurs}} (\mathcal{S}, \mathcal{F}, P(\cdot \mid M))$$

Recall...

13.2

$P(\cdot \mid M)$ is a valid prob. measure $\Rightarrow$ $F_{X}(x \mid M)$ is a valid cdf

$\therefore$ $F_{X}(x \mid M)$ has all the properties of a valid cdf.

e.g. $P(\{a < X \le b\} \mid M) = F_{X}(b \mid M) - F_{X}(a \mid M)$

Recall...

13.3

Defn: The conditional probability density function (conditional pdf) of X conditioned on $M \in \mathcal{F}$ is

$$f_{X}(x \mid M) \triangleq \frac{d F_{X}(x \mid M)}{dx}.$$

n.b. $F_{X}(x \mid M)$ is a valid cdf $\xRightarrow{\frac{d}{dx}}$ $f_{X}(x \mid M)$ is a valid pdf

e.g. $P(\{a < X \le b\} \mid M) = \int_a^b f_{X}(x \mid M)\, dx$

Recall...

In general, we must know the underlying structure of $(\mathcal{S}, \mathcal{F}, P)$ and the mapping $\mathbf{X}$ to determine $F_{\mathbf{X}}(x|M)$ or $f_{\mathbf{X}}(x|M)$.

But sometimes we can describe the event $M \in \mathcal{F}$ in terms of the RV $\mathbf{X}$.

e.g. 1. $M = \{\mathbf{X} \le a\}$, $a \in \mathbb{R}$.

2. $\boxed{M = \{b < \mathbf{X} \le a\}, \ a, b \in \mathbb{R}, \ b < a}$

2. $M = \{b < \mathbf{X} \le a\}$, $b < a$.

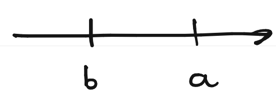
$$F_{\mathbf{X}}(x|M) = P(\{\mathbf{X} \le x\} | \{b < \mathbf{X} \le a\})$$
$$= \frac{P(\{\mathbf{X} \le x\} \cap \{b < \mathbf{X} \le a\})}{P(\{b < \mathbf{X} \le a\})}$$

Three distinct regions:

(a) $x > a$

(b) $b < x \le a$

(c) $x \le b$



13.6

Analyzing these 3 cases, we get
(exercise)

$$F_X(x|M) = \begin{cases} 1, & x > a \\ 0, & x \leq b \\ \dfrac{F_X(x) - F_X(b)}{F_X(a) - F_X(b)}, & b < x \leq a \end{cases}$$

The corresponding conditional pdf is

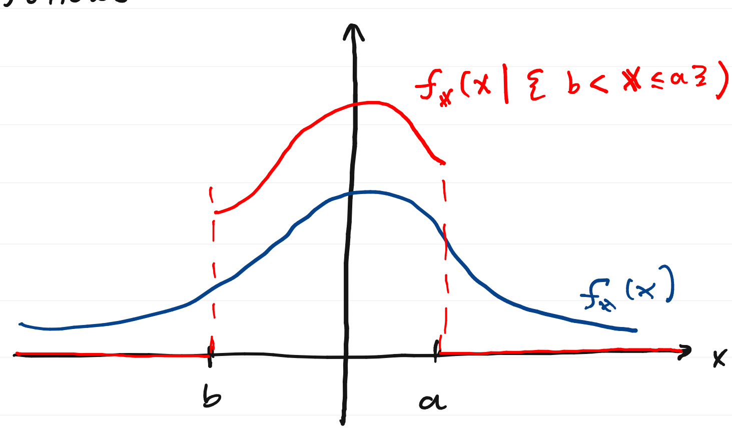
$$f_X(x|\{b < X \leq a\}) = \frac{dF_X(x|\{b < X \leq a\})}{dx}$$

$$= \begin{cases} 0, & x > a \\ 0, & x \leq b \\ \dfrac{f_X(x)}{F_X(a) - F_X(b)}, & b < x \leq a. \end{cases}$$

Recall...

13.7

A picture of these pdfs appears as follows:



Recall...

Now consider $P(A|B) = \frac{P(B|A)P(A)}{P(B)}$ 13.8

Let $B = \{x_1 < X \leq x_2\}$, $x_1 < x_2$

Then we have

$$P(A|\{x_1 < X \leq x_2\}) = \frac{P(\{x_1 < X \leq x_2\}|A)\,P(A)}{P(\{x_1 < X \leq x_2\})}$$

$$= \frac{(F_X(x_2|A) - F_X(x_1|A))\,P(A)}{F_X(x_2) - F_X(x_1)} \quad \text{---} \quad (3)$$

What about $P(A|\{X = x\})$ 13.9

Suppose $B = \{X = x\}$. what is $P(A|B)$?

$$P(A|\{X=x\}) = \frac{P(A \cap \{X = x\})}{P(\{X=x\})} = \frac{0}{0}$$

n.b. If X is continuous, $P(\{X=x\}) = 0$,
and $P(A \cap \{X=x\}) \leq P(\{X=x\})$

$\Rightarrow$ $P(A \cap \{X=x\}) = 0$.

So we have

$$P(A \mid \{X = x\}) = \frac{P(A \cap \{X = x\})}{P(\{X = x\})} = \frac{0}{0} = \text{undefined}$$

But clearly the probability

$$P(A \mid \{X = x\})$$

is meaningful.

Approach:

Consider $P(A \mid \{x_1 < X \leq x_2\})$

where $x_1 = x$

$x_2 = x + \Delta x$

Then we consider

$$\lim_{\Delta x \to 0} P(A \mid \{x < X \leq x + \Delta x\}) = P(A \mid \{X = x\})$$

So

$$P(A \mid \{X = x\}) = \lim_{\Delta x \to 0} P(A \mid \{x < X \le x + \Delta x\})$$

$$\overset{(3)}{=} \lim_{\Delta x \to 0} \left[\frac{F_X(x+\Delta x \mid A) - F_X(x \mid A)}{F_X(x+\Delta x) - F_X(x)} \right] P(A)$$

$$= \lim_{\Delta x \to 0} \left[\frac{\dfrac{F_X(x+\Delta x \mid A) - F_X(x \mid A)}{\Delta x}}{\dfrac{F_X(x+\Delta x) - F_X(x)}{\Delta x}} \right] P(A)$$

Aside:

Fact: $$\lim_{p \to 0} \frac{A(p)}{B(p)} = \frac{\left(\lim_{p \to 0} A(p)\right)}{\left(\lim_{q \to 0} B(q)\right)}$$

Assuming $\lim_{p \to 0} A(p)$, $\lim_{p \to 0} B(p)$ and $\lim_{p \to 0} \frac{A(p)}{B(p)}$ are well defined

$$= \cdots = \frac{f_X(x \mid A)}{f_X(x)} P(A).$$

13.14

$$\therefore \quad \boxed{P(A \mid \{X = x\}) = \frac{f_X(x \mid A)}{f_X(x)} P(A)} \quad \text{---} (4)$$

From (4), multiplying both sides by $f_X(x)$, we get

$$P(A \mid \{X=x\}) f_X(x) = f_X(x \mid A) P(A)$$

$$\Rightarrow \int_{-\infty}^{\infty} P(A \mid \{X=x\}) f_X(x)\, dx = P(A) \underbrace{\int_{-\infty}^{\infty} f_X(x \mid A)\, dx}_{=1}$$

$$\Rightarrow P(A) = \int_{-\infty}^{\infty} P(A \mid \{X=x\}) f_X(x)\, dx$$

13.15

$$\therefore \quad \boxed{P(A) = \int_{-\infty}^{\infty} P(A \mid \{X=x\}) f_X(x)\, dx} \quad \text{---} [5]$$

Total Probability Law

13.16

Furthermore from [4]

$$f_X(x|A) = \frac{P(A|\{X=x\})\, f_X(x)}{P(A)}$$

$$\therefore \boxed{f_X(x|A) = \frac{P(A|\{X=x\})\, f_X(x)}{\int_{-\infty}^{\infty} P(A|\{X=\alpha\})\, f_X(\alpha)\, d\alpha}}$$

Bayes Theorem --- [6]

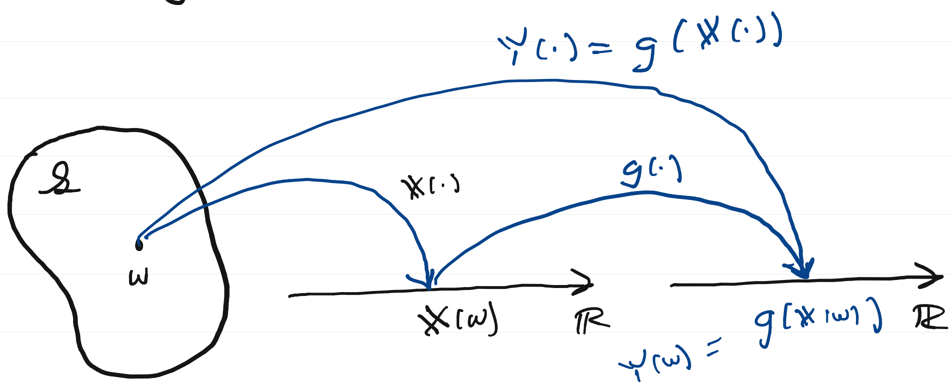
13.17

Functions of a Random Variable

Assume X is a RV on $(S, \mathcal{F}, P)$. Now assume we have a function

$$Y = g(X),$$

where $g : \mathbb{R} \to \mathbb{R}$



13.18

So $Y(\cdot) = g(X(\cdot)) : \mathcal{S} \to \mathbb{R}$.

It appears to be a random variable.

Is it?

Recall: From the defn. of a RV, $Y : \mathcal{S} \to \mathbb{R}$ is a RV if

$Y^{-1}(A) = \{\omega \in \mathcal{S} : Y(\omega) \in A\} \in \mathcal{F}, \ \forall A \in \mathcal{B}(\mathbb{R})$,

where $\mathcal{F}$ is the event space of $(\mathcal{S}, \mathcal{F}, P)$.

13.19

For $Y = g(X)$ to be measurable (i.e, an RV) $g(\cdot)$ must satisfy the following properties:

1. The domain of $g(\cdot)$ must contain the range space of X.

2. For each $y \in \mathbb{R}$, the set $R_y \triangleq \{x \in \mathbb{R} : g(x) \leq y\}$ must be a Borel set.

3. The events $\{g(X) = \pm\infty\}$ must have probability 0.

13.20

Any function $g(\cdot)$ satisfying these 3 properties is called a <u>Baire function</u>

For such functions $g(\cdot)$,

$$Y = g(X)$$

is a valid random variable.

All functions we typically encounter in engineering are Baire functions.

13.21

The Distribution of $Y = g(X)$

Given a RV X on $(S, \mathcal{F}, P)$ with cdf $F_X(x)$, and given a function $g: \mathbb{R} \to \mathbb{R}$, we define $Y = g(X)$.

Find

$$F_Y(y) = P(\{Y \le y\}) = P(\{g(X) \le y\})$$

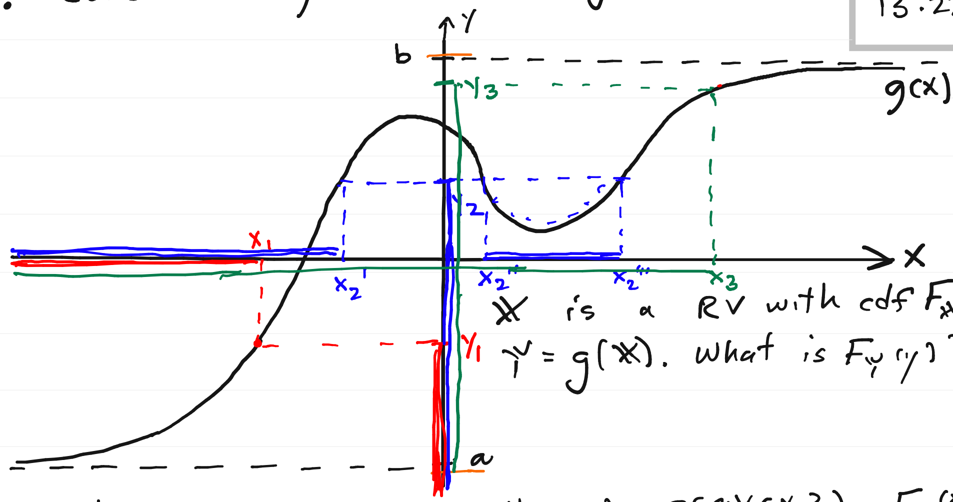
$$= P(\{X \in g^{-1}((-\infty, y])\})$$

$$= P_X(g^{-1}(-\infty, y])$$

This tells us how to find $F_Y(y)$ for any $y \in \mathbb{R}$.

13.22

1. Consider a generic function $g(\cdot)$:



X is a RV with cdf $F_X(x)$.
$Y = g(X)$. What is $F_Y(y)$?

Consider y_1: $F_Y(y_1) = P(\{Y \le y_1\}) = P(\{X \le x_1\}) = F_X(x_1)$

Consider y_2: $F_Y(y_2) = P(\{Y \le y_2\}) = P(\{X \le x_2'\} \cup \{x_2'' \le X \le x_2'''\})$

disjoint

$= F_X(x_2') + F_X(x_2''') - F_X(x_2'')$

Consider y_3: $F_Y(y_3) = \cdots = F_X(x_3)$

14.18

13.23

$\therefore F_Y(y_2) = P(\{X \le x_2'\}) + P(\{x_2'' \le X \le x_2'''\})$

$= F_X(x_2') + (F_X(x_2''') - F_X(x_2''))$

If X is not absolutely continuous, we have to think carefully about the situation at $x = x_2''$.

So to determine $F_Y(y)$ completely, we must do this for all $y \in \mathbb{R}$.

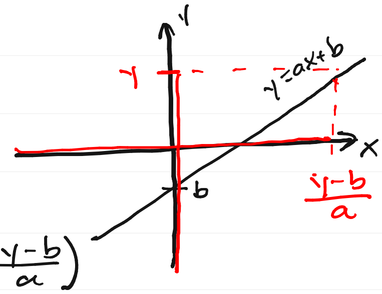
This can be easy or difficult depending the complexity of $g(\cdot)$

14.19

Ex. A $\quad Y = aX + b, \quad a, b \in \mathbb{R}$

$\Rightarrow \quad g(x) = ax + b$ (affine transformation)

two cases: $\quad a \gtrless 0$.



(i) $a > 0$

$$F_Y(y) = P(\{Y \le y\}) = P(\{aX + b \le y\})$$

$$= P\left(\left\{X \le \frac{y-b}{a}\right\}\right) = F_X\left(\frac{y-b}{a}\right)$$

$$f_Y(y) = \frac{dF_Y(y)}{dy} = f_X\left(\frac{y-b}{a}\right) \cdot \frac{1}{a}$$

$$= \frac{1}{a} f_X\left(\frac{y-b}{a}\right)$$

(ii) $a < 0$: $\quad F_Y(y) = P(\{Y \le y\})$

$$= P(\{aX + b \le y\})$$

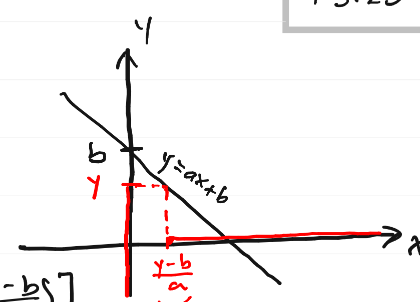
$$= P\left(\left\{X \ge \frac{y-b}{a}\right\}\right)$$

$$= P\left(\left\{X > \frac{y-b}{a}\right\}\right)$$

$$= 1 - F_X\left(\frac{y-b}{a}\right)$$

$$\Rightarrow f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{d}{dy}\left[1 - F_X\left(\frac{y-b}{a}\right)\right]$$

$$= -f_X\left(\frac{y-b}{a}\right) \cdot \frac{1}{a} = -\frac{1}{a} f_X\left(\frac{y-b}{a}\right)$$

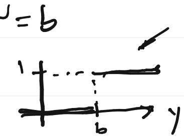


Combining case (i) and (ii), we get

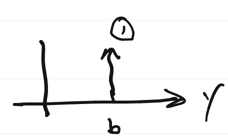
$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$

$a = 0$, $Y = b$

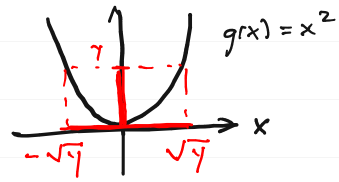
$F_Y(y)$



$f_Y(y)$



Ex. B: $Y = g(X) = X^2 \Rightarrow g(x) = x^2$.



Note immediately that

$$F_Y(y) = 0, \quad y < 0.$$

For $y > 0$:

$$F_Y(y) = P(\{Y \le y\}) = P(\{X^2 \le y\})$$
$$= P(-\sqrt{y} \le X \le \sqrt{y}\})$$
$$= P(\{-\sqrt{y} \le X \le \sqrt{y}\})$$
$$= F_X(\sqrt{y}) - F_X(-\sqrt{y})$$

$$\therefore \; F_Y(y) = [F_X(\sqrt{y}) - F_X(-\sqrt{y})] \, \mathbb{1}_{(0,\infty)}(y)$$