Session 13

Recall...

13.1

Ex. 1 Gaussian RV

A RV X is Gaussian if it has a pdf of the form

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left\{ -\frac{(x-\mu)^2}{2\sigma^2} \right\}, \forall x \in \mathbb{R}$$

where $\mu \in \mathbb{R}$ and $\sigma > 0$.

n.b. $F_X(x) = \int_{-\infty}^{x} f_X(\alpha)\, d\alpha = \Phi\left(\frac{x-\mu}{\sigma}\right)$ $\left(G\left(\frac{x-\mu}{\sigma}\right) \text{ in Papoulis} \right)$

$\mathcal{N}[\mu, \sigma^2]$

where $\Phi(r) = \int_{-\infty}^{r} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz$.

Recall...

Ex. 2 Uniformly Distributed RV

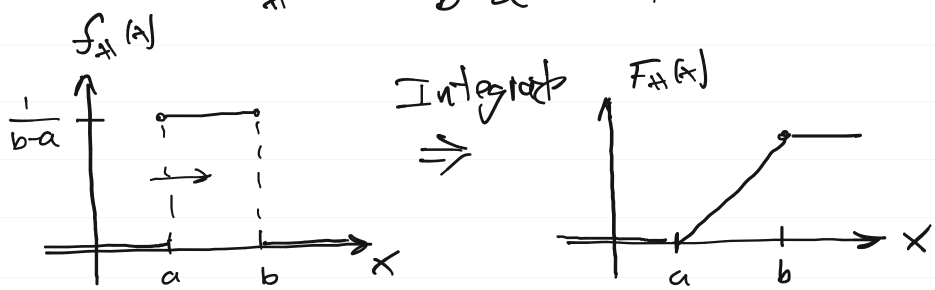
13.2

A RV has a uniform distribution,

$\mathbb{X} \sim U[a,b]$, $a<b$

if

$$f_{\mathbb{X}}(x) = \frac{1}{b-a} \cdot 1_{[a,b]}(x)$$



Ex. 3 Binomially Distributed RV

13.3

A binomially distributed RV is a discrete RV taking on values in the set $\{0,1,2,\dots,n\} \subset \mathbb{R}$ with pmf

$$P_{\mathbb{X}}(k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k=0,1,2,\dots,n, \quad p \in [0,1]$$

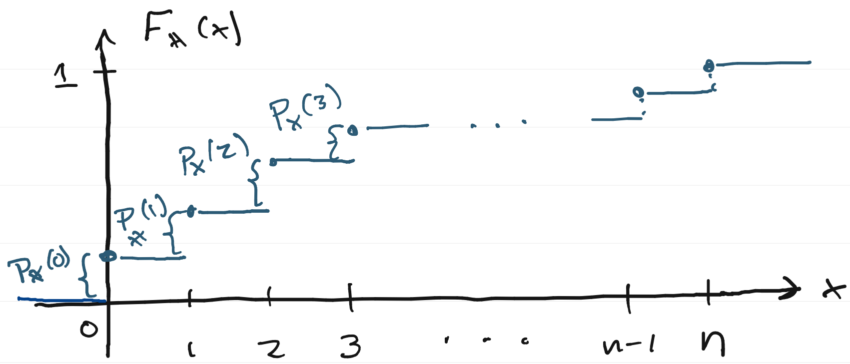
The cdf of this RV is

$$F_{\mathbb{X}}(x) = P(\{\mathbb{X} \le x\}) = \sum_{k=0}^{m(x)} \binom{n}{k} p^k (1-p)^{n-k}$$

where $m(x)$ is an integer such that

$m(x) \le x < m(x)+1$.

13.4



$$f_X(x) = \text{"}\frac{dF_X(x)}{dx}\text{"} = \sum_{k=0}^{n} P_X(k)\,\delta(x-k)$$

$$= \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k}\,\delta(x-k)$$

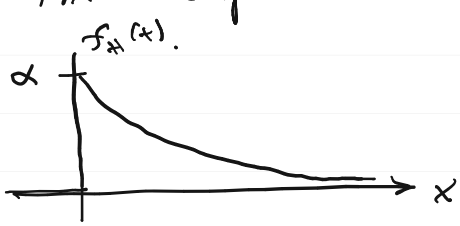
13.5

Ex.4 Exponentially Distributed RV

A RV X with a pdf of the form

$$f_X(x) = \alpha e^{-\alpha x} \cdot 1_{[0,\infty)}(x) = \begin{cases} \alpha e^{-\alpha x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

where $\alpha > 0$ is called the parameter of the exponential RV.



See examples of other types of RVs in Papoulis

You should know the following pdfs and pmfs:

pmfs	pdfs
Binomial	Gaussian
Poisson	Exponential
Geometric	Uniform

Conditional Distributions

Given $(\mathcal{S}, \mathcal{F}, P)$ with a RV X defined on it, and given $A, M \in \mathcal{F}$, we know that

$$P(A|M) = \frac{P(A \cap M)}{P(M)}, \quad P(M) > 0.$$

Take $A =$ "$\{X \le x\}$" $= \{\omega \in \mathcal{S} : X(\omega) \le x\}$

Then $P(A|M) = P(\{X \le x\} | M)$

This is a conditional cdf of X conditioned on the event $M \in \mathcal{F}$.

Defn: The conditional cdf of the RV X conditioned on $M \in \mathcal{F}$ is

$$F_X(x|M) = P(\{X \le x\}|M)$$
$$= \frac{P(\{X \le x\} \cap M)}{P(M)}, \quad x \in \mathbb{R}$$

The defn. of $F_X(x|M)$ is just like the defn. of $F_X(x)$, except we use the conditional prob. measure $P(\cdot|M)$ instead of $P(\cdot)$.

$P(\cdot|M)$ is a valid prob. measure $\Rightarrow$ $F_X(x|M)$ is a valid cdf

$\therefore$ $F_X(x|M)$ has all the properties of a valid cdf.

e.g. $P(\{a < X \le b\}|M) = F_X(b|M) - F_X(a|M)$

13.10

Defn: The <u>conditional probability density function</u> (conditional pdf) of $\mathbb{X}$ conditioned on $M \in \mathcal{F}$ is

$$f_{\mathbb{X}}(x|M) \triangleq \frac{dF_{\mathbb{X}}(x|M)}{dx}.$$

<u>n.b.</u> $F_{\mathbb{X}}(x|M)$ is a valid cdf $\xRightarrow{\frac{d}{dx}}$ $f_{\mathbb{X}}(x|M)$ is a valid pdf

e.g. $P(\{a < \mathbb{X} \le b\} | M) = \int_a^b f_{\mathbb{X}}(x|M)\, dx$

13.11

In general, we must know the underlying structure of $(\mathcal{S}, \mathcal{F}, P)$ and the mapping $\mathbb{X}$ to determine $F_{\mathbb{X}}(x|M)$ or $f_{\mathbb{X}}(x|M)$.

But sometimes we can describe the event $M \in \mathcal{F}$ in terms of the RV $\mathbb{X}$.

<u>e.g.</u>

1. $M = \{\mathbb{X} \le a\}$, $a \in \mathbb{R}$.

2. $M = \{b < \mathbb{X} \le a\}$, $a, b \in \mathbb{R}$, $b < a$

13.12

$\underline{1.}$ Let $M = \{ \mathbb{X} \le a \}$, $a \in \mathbb{R}$

$$F_{\mathbb{X}}(x|M) = P(\{\mathbb{X} \le x\} | \{\mathbb{X} \le a\})$$

$$= \frac{P(\{\mathbb{X} \le x\} \cap \{\mathbb{X} \le a\})}{P(\{\mathbb{X} \le a\})}$$

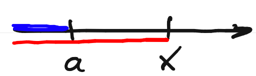
<u>Two cases:</u> $x > a$ and $x \le a$.

13.13

1. Let $M = \{ \mathbb{X} \le a \}$

(a) $x > a$

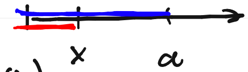
$$\underline{\{\mathbb{X} \le x\}} \cap \underline{\{\mathbb{X} \le a\}} = \{\mathbb{X} \le a\}$$



$$F_{\mathbb{X}}(x|M) = \frac{P(\{\mathbb{X} \le a\})}{P(\{\mathbb{X} \le a\})} = \frac{F_{\mathbb{X}}(a)}{F_{\mathbb{X}}(a)} = 1$$

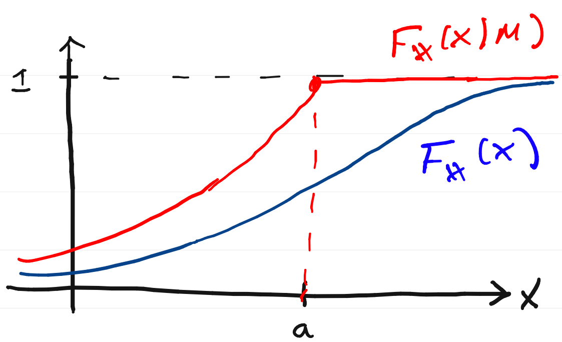
(b): <u>If $x \le a$:</u>

$$\{X \le x\} \cap \{X \le a\} = \{X \le x\}$$



$$F_X(x/M) = \frac{P(\{X \le x\})}{P(\{X \le a\})} = \frac{F_X(x)}{F_X(a)}$$

$$\therefore \quad F_X(x|M) = \begin{cases} \dfrac{F_X(x)}{F_X(a)}, & x \le a \\ 1, & x > a \end{cases}$$

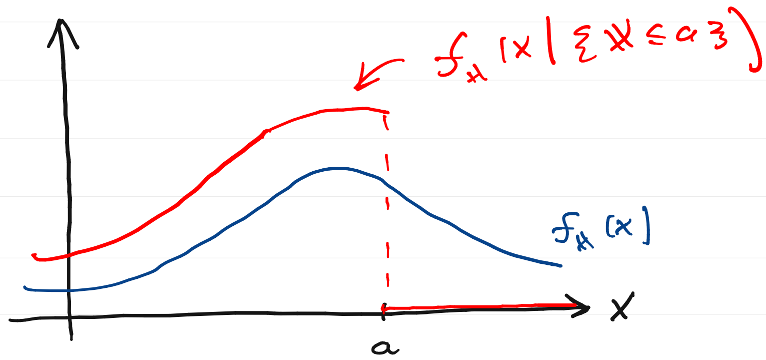


<u>Conditional pdf:</u>

$$f_X(x|M) = \frac{dF_X(x|\{X \le a\})}{dx}$$

$$= \begin{cases} \dfrac{f_X(x)}{F_X(a)}, & x \le a \\ 0, & x > a \end{cases}$$

13.16



13.17

2. $M = \{b < X \le a\}$, $b < a$.

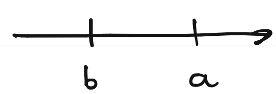
$$F_X(x \mid M) = P(\{X \le x\} \mid \{b < X \le a\})$$
$$= \frac{P(\{X \le x\} \cap \{b < X \le a\})}{P(\{b < X \le a\})}$$

Three distinct regions:

(a) $x > a$

(b) $b < x \le a$

(c) $x \le b$



Analyzing these 3 cases, we get
(exercise)

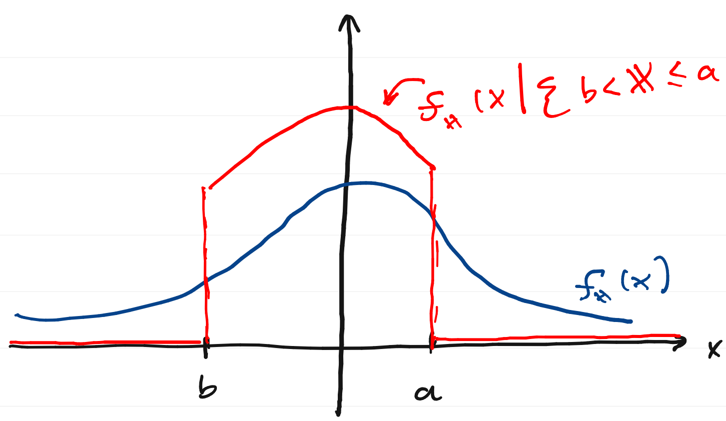
$$F_X(x|M) = \begin{cases} 1, & x > a \\ 0, & x \le b \\ \dfrac{F_X(x) - F_X(b)}{F_X(a) - F_X(b)}, & b < x \le a \end{cases}$$

The corresponding conditional pdf is

$$f_X(x|\{b < X \le a\}) = \frac{dF_X(x|\{b < X \le a\})}{dx}$$

$$= \begin{cases} 0, & x > a \\ 0, & x \le b \\ \dfrac{f_X(x)}{F_X(a) - F_X(b)}, & b < x \le a \end{cases}$$

A picture of these pdfs appears as follows:



The Total Prob. Law and Bayes Theorem

13.20

Given a RV X on $(S, \mathcal{F}, P)$, let $\{A_1, \ldots, A_n\}$ be a partition of S, with $A_k \in \mathcal{F}$, $k = 1, \ldots, n$.

Then

$$P(\{X \le x\}) = P(\{X \le x\} | A_1) P(A_1) + P(\{X \le x\} | A_2) P(A_2) + \cdots + P(\{X \le x\} | A_n) P(A_n) \quad \text{--- (1)}$$

But note that

$$P(\{X \le x\}) = F_X(x)$$

$$P(\{X \le x\} | A_k) = F_X(x | A_k)$$

13.21

$$\therefore \quad F_X(x) = F_X(x|A_1) P(A_1) + F_X(x|A_2) P(A_2) + \cdots + F_X(x|A_n) P(A_n) \quad \text{--- (1A)}$$

<u>note</u> $f_X(x) = \dfrac{dF_X(x)}{dx}$

and $f_X(x|A_k) = \dfrac{dF_X(x|A_k)}{dx}$

$$\Rightarrow \quad f_X(x) = f_X(x|A_1) P(A_1) + f_X(x|A_2) P(A_2) + \cdots + f_X(x|A_n) P(A_n) \quad \text{--- (1B)}$$

13.22

2. Recall Bayes Formula:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}.$$

Let $B = \{X \le x\}$. Then

$$P(A|\{X \le x\}) = \frac{P(\{X \le x\}|A)P(A)}{P(\{X \le x\})}$$

$$= \frac{F_X(x|A)P(A)}{F_X(x)}$$

$$\therefore \quad P(A|\{X \le x\}) = \frac{F_X(x|A)P(A)}{F_X(x)}$$