Session 11

Exam 1: Thursday, Feb. 20, 2025

- On-Campus: 10:30 – 11:45 am, (Here) FNY B124
- On-Campus Online Overflow: 10:30 – 11:45 am WALC 2127
- Online EPE: See Lynn Hegewald's Instructions.

Same format as the Sample Exam:

- Closed book, Closed notes, No calculators, No crib sheets
- 4 problems worth 25% each.

Coverage: Beginning of class through slide 10.15.
Homework through Problem 6 of Hwk. 4.

Recall ...

We often describe a RV $\mathbb{X}$ by specifying its cdf or pdf and completely ignoring the underlying $(\mathcal{S}, \mathcal{F}, P)$.

(The underlying $(\mathcal{S}, \mathcal{F}, P)$ is there, but we just don't think about it.)

$$(\mathcal{S}, \mathcal{F}, P) \xRightarrow{\mathbb{X}(\cdot)} (\mathbb{R}, \mathcal{B}(\mathbb{R}), P_X)$$

Ex. 1 Gaussian RV

A RV $\mathbb{X}$ is Gaussian if it has a pdf of the form

$$f_{\mathbb{X}}(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left\{ \frac{-(x-\mu)^2}{2\sigma^2} \right\}, \quad \forall x \in \mathbb{R},$$

where $\mu \in \mathbb{R}$ and $\sigma > 0$.

n.b. $$F_{\mathbb{X}}(x) = \int_{-\infty}^{x} f_{\mathbb{X}}(\alpha)\, d\alpha = \Phi\left(\frac{x-\mu}{\sigma}\right) = G\left(\frac{x-\mu}{\sigma}\right)$$

Papoulis

$$\Phi(r) \triangleq \int_{-\infty}^{r} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz$$

$\Phi(\cdot)$ cannot be written in closed form. It can be numerically computed and is widely tabulated.

So if X is a Gaussian RV with $\mu \in \mathbb{R}$ and $\sigma > 0$, then

$$P(\{a < X \le b\}) = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$$

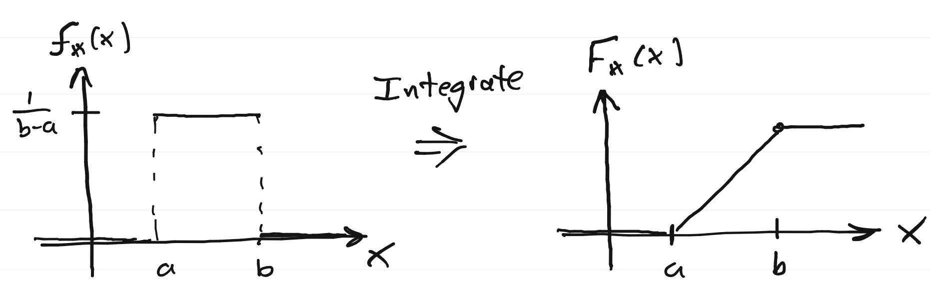
n.b. $F_X(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$.

Ex.2 Uniformly Distributed RV

A RV has a uniform distribution,

$$X \sim U[a,b], \quad a<b$$

if $f_X(x) = \frac{1}{b-a} \mathbb{1}_{[a,b]}(x)$



Ex. 3 Binomially Distributed RV

11.5

A binomially distributed RV is a discrete RV taking on values in the set $\{0, 1, 2, \ldots, n\} \subset \mathbb{R}$ with pmf

$$P_X(k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, 2, \ldots, n, \quad p \in [0,1]$$

The cdf of this RV is

$$F_X(x) = P(\{X \le x\}) = \sum_{k=0}^{m(x)} \binom{n}{k} p^k (1-p)^{n-k}$$

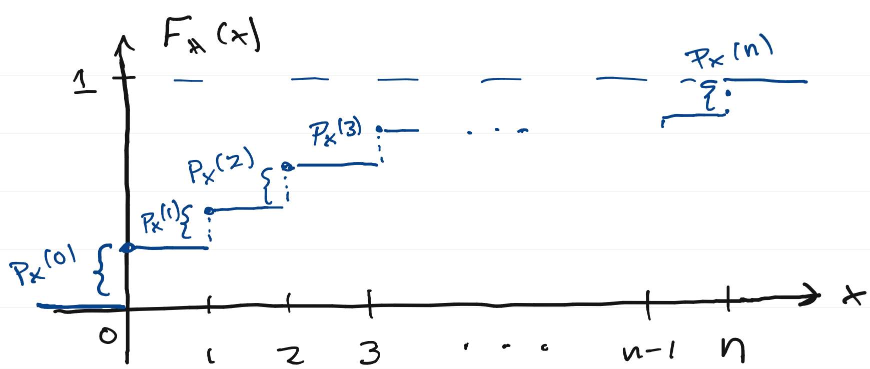
where $m(x)$ is the integer such that

$$m(x) \le x < m(x) + 1$$

$m(x) = [x]$ integer part of x

13-3

11.6

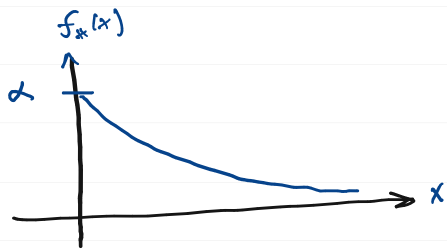


Ex. 4 Exponentially Distributed RV

A RV X with a pdf of the form

$$f_X(x) = \alpha e^{-\alpha x} \cdot 1_{[0,\infty)}(x) = \begin{cases} \alpha e^{-\alpha x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

where $\alpha > 0$ is called the parameter of the exponential RV.



See examples of other types of RVs in Papoulis

You should know the following pdfs and pmfs:

pmfs	pdfs
Binomial	Gaussian
Poisson	Exponential
Geometric	Uniform

$\boxed{\text{Complete Hwk 4 Problems 1–6.}}$

Cut-off for Exam 1

Conditional Distributions

11.9

Given $(\mathcal{S}, \mathcal{F}, P)$ with a RV $\mathbb{X}$ defined on it, and given $A, M \in \mathcal{F}$, we know that

$$P(A|M) = \frac{P(A \cap M)}{P(M)}, \quad P(M) > 0.$$

Take $A = $ "$\{\mathbb{X} \le x\}$" $= \{\omega \in \mathcal{S} : \mathbb{X}(\omega) \le x\}$

Then $P(A|M) = P(\{\mathbb{X} \le x\} | M)$

This is a conditional cdf of $\mathbb{X}$ conditioned on the event $M \in \mathcal{F}$.

11.10

Defn: The conditional cdf of the RV $\mathbb{X}$ conditioned on $M \in \mathcal{F}$ is

$$F_{\mathbb{X}}(x|M) = P(\{\mathbb{X} \le x\} | M)$$

$$= \frac{P(\{\mathbb{X} \le x\} \cap M)}{P(M)}, \quad x \in \mathbb{R}$$

The definition of $F_{\mathbb{X}}(x|M)$ is just like the definition of $F_{\mathbb{X}}(x)$, except we use the conditional probability measure $P(\cdot|M)$ instead of $P(\cdot)$.

$$(\mathcal{S}, \mathcal{F}, P) \xRightarrow{M \text{ occurs}} (\mathcal{S}, \mathcal{F}, \underline{P(\cdot|M)})$$

$P(\cdot \mid M)$ is a valid prob. measure $\Rightarrow$ $F_{X}(x \mid M)$ is a valid cdf

$\therefore$ $F_{X}(x \mid M)$ has all the properties of a valid cdf.

e.g. $P(\{a < X \leq b\} \mid M) = F_{X}(b \mid M) - F_{X}(a \mid M)$

Defn: The conditional probability density function (conditional pdf) of X conditioned on $M \in \mathcal{F}$ is

$$f_{X}(x \mid M) \triangleq \frac{d F_{X}(x \mid M)}{dx}.$$

n.b. $F_{X}(x \mid M)$ is a valid cdf $\overset{\frac{d}{dx}}{\Longrightarrow}$ $f_{X}(x \mid M)$ is a valid pdf

e.g. $P(\{a < X \leq b\} \mid M) = \int_a^b f_{X}(x \mid M)\, dx$

In general, we must know the underlying structure of $(\mathcal{S}, \mathcal{F}, P)$ and the mapping $\mathbb{X}$ to determine $F_{\mathbb{X}}(x|M)$ or $f_{\mathbb{X}}(x|M)$.

But sometimes we can describe the event $M \in \mathcal{F}$ in terms of the RV $\mathbb{X}$.

e.g. 1. $M = \{\mathbb{X} \le a\}$, $a \in \mathbb{R}$.

2. $M = \{b < \mathbb{X} \le a\}$, $a, b \in \mathbb{R}$, $b <$

1. Let $M = \{\mathbb{X} \le a\}$, $a \in \mathbb{R}$

$$F_{\mathbb{X}}(x|M) = P(\{\mathbb{X} \le x\} \mid \{\mathbb{X} \le a\})$$
$$= \frac{P(\{\mathbb{X} \le x\} \cap \{\mathbb{X} \le a\})}{P(\{\mathbb{X} \le a\})}$$

Two cases: $x > a$ and $x \le a$

1. Let $M = \{\mathbf{X} \le a\}$

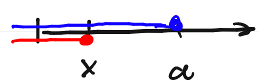
(a) $x > a$

$$\{\mathbf{X} \le x\} \cap \{\mathbf{X} \le a\} = \{\mathbf{X} \le a\}$$

$$F_{\mathbf{X}}(x|M) = \frac{P(\{\mathbf{X} \le a\})}{P(\{\mathbf{X} \le a\})} = \frac{F_{\mathbf{X}}(a)}{F_{\mathbf{X}}(a)} = 1.$$

(b): If $x \le a$:

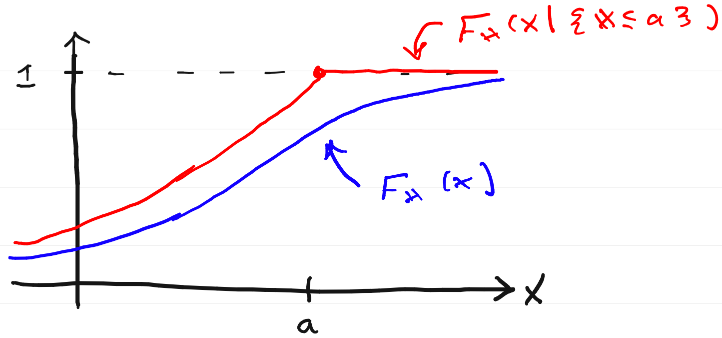
$$\{\mathbf{X} \le x\} \cap \{\mathbf{X} \le a\} = \{\mathbf{X} \le x\}$$



$$F_{\mathbf{X}}(x/M) = \frac{P(\{\mathbf{X} \le x\})}{P(\{\mathbf{X} \le a\})} = \frac{F_{\mathbf{X}}(x)}{F_{\mathbf{X}}(a)}$$

$$\therefore \quad F_{\mathbf{X}}(x/M) = \begin{cases} \dfrac{F_{\mathbf{X}}(x)}{F_{\mathbf{X}}(a)}, & x \le a \\ 1, & x > a. \end{cases}$$

11.17

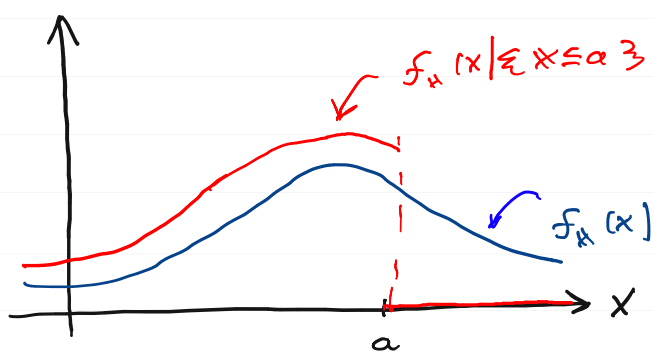


<u>Conditional pdf:</u>

$$f_X(x \mid M) = \frac{dF_X(x \mid \{X \le a\})}{dx}$$

$$= \begin{cases} \dfrac{f_X(x)}{F_X(a)}, & x \le a \\ 0, & x > a \end{cases}$$

11.18



2. $M = \{ b < \mathbb{X} \le a \}$, $b < a$.

$$F_{\mathbb{X}}(x|M) = P(\{\mathbb{X} \le x\} | \{ b < \mathbb{X} \le a \})$$

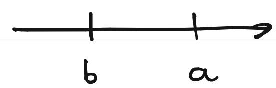
$$= \frac{P(\{\mathbb{X} \le x\} \cap \{ b < \mathbb{X} \le a \})}{P(\{ b < \mathbb{X} \le a \})}$$

Three distinct regions:

(a) $x > a$

(b) $b < x \le a$

(c) $x \le b$



Analyzing these 3 cases, we get
(exercise)

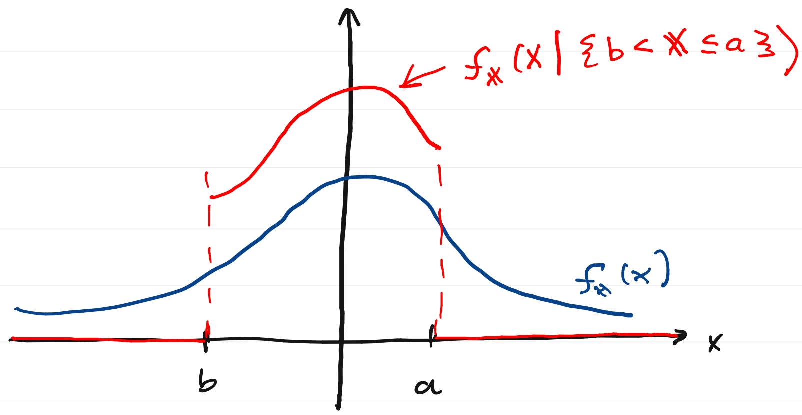
$$F_{\mathbb{X}}(x|M) = \begin{cases} 1, & x > a \\ 0, & x \le b \\ \dfrac{F_{\mathbb{X}}(x) - F_{\mathbb{X}}(b)}{F_{\mathbb{X}}(a) - F_{\mathbb{X}}(b)}, & b < x \le a \end{cases}$$

The corresponding conditional pdf is

$$f_{\mathbb{X}}(x | \{ b < \mathbb{X} \le a \}) = \frac{d F_{\mathbb{X}}(x | \{ b < \mathbb{X} \le a \})}{dx}$$

$$= \begin{cases} 0, & x > a \\ 0, & x \le b \\ \dfrac{f_{\mathbb{X}}(x)}{F_{\mathbb{X}}(a) - F_{\mathbb{X}}(b)}, & b < x \le a. \end{cases}$$

A picture of these pdfs appears as follows:



The Total Prob. Law and Bayes Theorem

Given a RV X on $(S, \mathcal{F}, P)$, let $\{A_1, \dots, A_n\}$ be a partition of S, with $A_k \in \mathcal{F}$, $k = 1, \dots, n$.

Then

$$P(\{X \le x\}) = P(\{X \le x\} \mid A_1) P(A_1) + P(\{X \le x\} \mid A_2) P(A_2) + \dots + P(\{X \le x\} \mid A_n) P(A_n) \quad \text{--- (1)}$$

But note that

$$P(\{X \le x\}) = F_X(x)$$

$$P(\{X \le x\} \mid A_k) = F_X(x \mid A_k)$$

11.23

$$\therefore \quad F_X(x) = F_X(x|A_1)P(A_1) + F_X(x|A_2)P(A_2) + \cdots + F_X(x|A_n)P(A_n) \quad \text{--- (1A)}$$

note $\quad f_X(x) = \dfrac{dF_X(x)}{dx}$

and $\quad f_X(x|A_k) = \dfrac{dF_X(x|A_k)}{dx}$

$\Rightarrow$

$$f_X(x) = f_X(x|A_1)P(A_1) + f_X(x|A_2)P(A_2) + \cdots + f_X(x|A_n)P(A_n) \quad \text{--- (1B)}$$

11.24

2. Recall Bayes Formula:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}.$$

Let $B = \{X \le x\}$. Then

$$P(A|\{X \le x\}) = \frac{P(\{X \le x\}|A)P(A)}{P(\{X \le x\})} = \frac{F_X(x|A)P(A)}{F_X(x)}$$

$$\therefore \quad P(A|\{X \le x\}) = \frac{F_X(x|A)P(A)}{F_X(x)}$$

11.25

Now consider $P(A|B) = \dfrac{P(B|A)\,P(A)}{P(B)}$

Let $B = \{x_1 < \mathbb{X} \le x_2\}$, $x_1 < x_2$

Then we have

$$P(A \mid \{x_1 < \mathbb{X} \le x_2\}) = \frac{P(\{x_1 < \mathbb{X} \le x_2\} \mid \underline{A})\,P(A)}{P(\{x_1 < \mathbb{X} \le x_2\})}$$

$$= \frac{\left(F_{\mathbb{X}}(x_2|A) - F_{\mathbb{X}}(x_1|A)\right) P(A)}{F_{\mathbb{X}}(x_2) - F_{\mathbb{X}}(x_1)} \quad \text{---} \quad (3)$$

11.26

What about $P(A \mid \{\mathbb{X} = x\})$

Suppose $B = \{\mathbb{X} = x\}$. what is $P(A|B)$?

$$P(A \mid \{\mathbb{X} = x\}) = \frac{\overbrace{P(A \cap \{\mathbb{X} = x\})}^{=0}}{P(\{\mathbb{X} = x\}) = 0} = \frac{0}{0}$$

n.b. If $\mathbb{X}$ is continuous, $P(\{\mathbb{X} = x\}) = 0$,
and $P(A \cap \{\mathbb{X} = x\}) \le P(\{\mathbb{X} = x\})$

$\Rightarrow$ $P(A \cap \{\mathbb{X} = x\}) = 0$.