

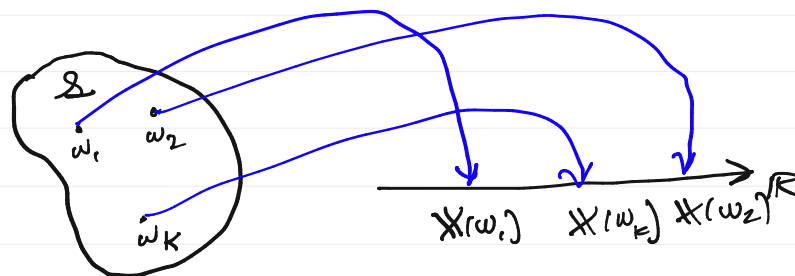
Session 11

Recall ...

Intuitive "Defn": Given (Ω, \mathcal{F}, P) ,
a random variable is a mapping
from Ω to the real line.

11.1

$$X : \Omega \rightarrow \mathbb{R}$$



Recall...

11.2

Defn: Given (Ω, \mathcal{F}, P) a random variable is a mapping $X: \Omega \rightarrow \mathbb{R}$ with the property that for all $A \in \mathcal{B}(\mathbb{R})$,
 $X^{-1}(A) = \{\omega \in \Omega : X(\omega) \in A\} \in \mathcal{F}$

Such a function is called a Borel measurable function.

8.19

Recall...

11.3

The "Range Space" of a function f

Let $f: \Omega \rightarrow A$.

We call the image $f(\Omega) \subset A$ the range space of f .

e.g. $f: \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = x^2$

then " $f(\mathbb{R})$ " = $[0, \infty)$ = range space of $f(x) = x^2$.

Given (Ω, \mathcal{F}, P) and a R.V. X , the R.V. takes on values in the range space
 $\mathcal{C} = f(\Omega) \subset \mathbb{R}$.

Because X is measurable, all subsets of the form

11.4

$$X^{-1}(G) = \{\omega \in \Omega : X(\omega) \in G\}$$

where $G \in \mathcal{B}(\mathcal{E})$

must be in the event space \mathcal{F} of (Ω, \mathcal{F}, P) .

So we can compute the probability $P_X(G)$ that $X(\omega)$ takes on a value in $G \in \mathcal{B}(\mathcal{E})$.

10.12

Here

11.5

$$P_X(G) = P(X^{-1}(G))$$

$$= P(\{\omega \in \Omega : X(\omega) \in G\}), \forall G \in \mathcal{B}(\mathcal{E}).$$

So we have the situation

$$(\Omega, \mathcal{F}, P) \xrightarrow{X(\cdot)} (\mathcal{E}, \mathcal{B}(\mathcal{E}), P_X).$$

So we can focus on the new random experiment $(\mathcal{E}, \mathcal{B}(\mathcal{E}), P_X)$.

Good News: This is a probability space, and we know how to deal with probability spaces.

10.13

Q: So why do we describe a
RV as a mapping in the
first place?

- A:
1. Insight
 2. We can define multiple
RVs on one experiment
 3. Generalization to
Stochastic Processes.

Examples:

Ex. 1: Let $(\mathcal{S}, \mathcal{F}, P)$ be any prob.
space. Let $A \in \mathcal{F}$ be any
event.

$$\text{Define } \mathbb{1}_A(\omega) = \begin{cases} 1, & \omega \in A \\ 0, & \omega \notin A \end{cases}$$

Ex. 2 Suppose I toss a die.
that has six colored sides.

$$\mathcal{S} = \{ Br, R, O, Y, G, B \}$$

$$\mathcal{F} = \mathcal{P}(\mathcal{S})$$

Define a RV X as follows:

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$$\left. \begin{array}{l} X(B_1) = 1 \\ X(R) = 2 \\ \vdots \\ X(B_1) = 6 \end{array} \right\}$$

This is a measurable RV.

Ex. 3 Select a resistor at random from a box of resistors and measure its value.

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So we can think of a random variable in two ways:

1. Mathematically: A measurable function from $(\mathcal{S}, \mathcal{F}, P)$ to \mathbb{R} .
2. Intuitively: Something that takes on real values at random.

Given a random variable with range space $\mathcal{E} \subset \mathbb{R}$, we can construct a probability space $(\mathcal{E}, \mathcal{B}(\mathcal{E}), \mathcal{P}_x)$. 11.10

How do we describe \mathcal{P}_x ?

Discrete \mathcal{E} : We can use a pmf
$$\mathcal{P}_x(x) = \mathcal{P}_x(\{x\}), \forall x \in \mathcal{E}.$$

The outcomes of a random experiment $(\mathcal{E}, \mathcal{B}(\mathcal{E}), \mathcal{P}_x)$, $\mathcal{E} \subset \mathbb{R}$.

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Continuous (uncountable) \mathcal{E} : We can use a p.d.f. $f_x(x)$ to assign 11.11

$$\mathcal{P}_x(G) = \int_G f_x(x) dx, \forall G \in \mathcal{B}(\mathcal{E})$$

recall: Since $f_x(x)$ is a pdf:

$$(i) f_x(x) \geq 0, \forall x \in \mathcal{E}$$

$$(ii) \int_{\mathcal{E}} f_x(x) dx = 1.$$

Defn: Given a random variable X defined on $(\Omega, \mathcal{F}, \mathbb{P})$ that induces a new probability space $(\mathcal{E}, \mathcal{B}(\mathcal{E}), \mathbb{P}_X)$, the cumulative distribution function (cdf) of X is defined as

11.12

$$\begin{aligned} F_X(\alpha) &\triangleq \mathbb{P}_X((-\infty, \alpha]) = \mathbb{P}_X(\{x : x \leq \alpha\}) \\ &= \mathbb{P}(\{\omega \in \Omega : X(\omega) \leq \alpha\}) \\ &= \mathbb{P}(X^{-1}((-\infty, \alpha])), \quad \alpha \in \mathbb{R}. \end{aligned}$$

Notation: We often write

$$\{\{X \leq \alpha\}\}$$

$$\text{or } \mathbb{P}(\{X \leq \alpha\})$$

We understand that

$$\{\{X \leq \alpha\}\} = \{\omega \in \Omega : X(\omega) \leq \alpha\} \in \mathcal{F}.$$

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$F_x(\alpha)$ specifies $P_x(\cdot)$:

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Fact: It is possible to construct any set $F \in \mathcal{B}(\mathbb{R})$ using a countable sequence of set operations on the intervals of the form $(-\infty, x_n]$.

n.b. $F_x(x_n) = P_x((-\infty, x_n])$

n.b. We could have defined $\mathcal{B}(\mathbb{R})$ as the smallest σ -field containing all intervals of the form $(-\infty, x]$, $x \in \mathbb{R}$.
Instead we used the open sets.

So the cdf is defined as

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$$\begin{aligned} F_x(x) &= P(\{X \leq x\}), \quad \forall x \in \mathbb{R} \\ &= P(\{\omega \in \Omega : X(\omega) \leq x\}), \quad \forall x \in \mathbb{R} \\ &= P_x((-\infty, x]), \quad \forall x \in \mathbb{R}. \end{aligned}$$

Properties of the cdf:

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$$1. F_X(+\infty) = 1 \quad \text{and} \quad F_X(-\infty) = 0.$$

$$\text{n.b. } F_X(+\infty) = P(\underbrace{\{X \leq +\infty\}}_{\text{axiom 2}}) = 1$$

$$F_X(-\infty) = P(\underbrace{\{X \leq -\infty\}}_{\emptyset}) = 0.$$

2. If $x_1 < x_2$, then

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$$F_X(x_1) \leq F_X(x_2).$$

$$\text{n.b. } F_X(x_1) = P_X((-\infty, x_1])$$

$$F_X(x_2) = P_X((-\infty, x_2])$$

$$(-\infty, x_1] \subset (-\infty, x_2]$$

$$\Rightarrow (-\infty, x_2] = (-\infty, x_1] \cup \underbrace{(x_1, x_2]}_{\text{disjoint}}$$

$$P_X((-\infty, x_2]) = P_X((-\infty, x_1]) + \underbrace{P_X((x_1, x_2])}_{\geq 0}$$

$$\Downarrow \qquad \Downarrow$$
$$F_X(x_2) = F_X(x_1)$$

$$\therefore F_X(x_2) \geq F_X(x_1).$$

$$\underline{3.} \quad P(\{X > \alpha\}) = 1 - F_X(\alpha).$$

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$$\{X > \alpha\} = \overline{\{X \leq \alpha\}}$$

$$\Rightarrow P(\{X > \alpha\}) = 1 - P(\{X \leq \alpha\}) = 1 - F_X(\alpha).$$

4. If $x_1 < x_2$, then

$$P(\{x_1 < X \leq x_2\}) = F_X(x_2) - F_X(x_1).$$

n.b. $(-\infty, x_2] = (-\infty, x_1] \cup (x_1, x_2]$

\Downarrow

\uparrow disjoint \uparrow

$$P(\{-\infty < X \leq x_2\}) = P(\{-\infty < X \leq x_1\}) + P(\{x_1 < X \leq x_2\})$$

\Downarrow

\Downarrow

$$F_X(x_2) = F_X(x_1) + P(\{x_1 < X \leq x_2\})$$

$$\Rightarrow P(\{x_1 < X \leq x_2\}) = F_X(x_2) - F_X(x_1).$$

$$\underline{5.} \quad P(\{X = x_0\}) = F_X(x_0) - F_X(x_0^-)$$

11.19

where

$$F_X(x_0^-) = \lim_{\varepsilon \downarrow 0} F_X(x_0 - \varepsilon).$$

n.b. $F_X(x_0) - F_X(x_0^-)$

$$= P_X((-\infty, x_0]) - \lim_{\varepsilon \downarrow 0} P_X((-\infty, x_0 - \varepsilon])$$

$$= \lim_{\varepsilon \downarrow 0} P_X((x_0 - \varepsilon, x_0]) = P_X(\{x_0\})$$

In Summary:

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Properties of the cdf:

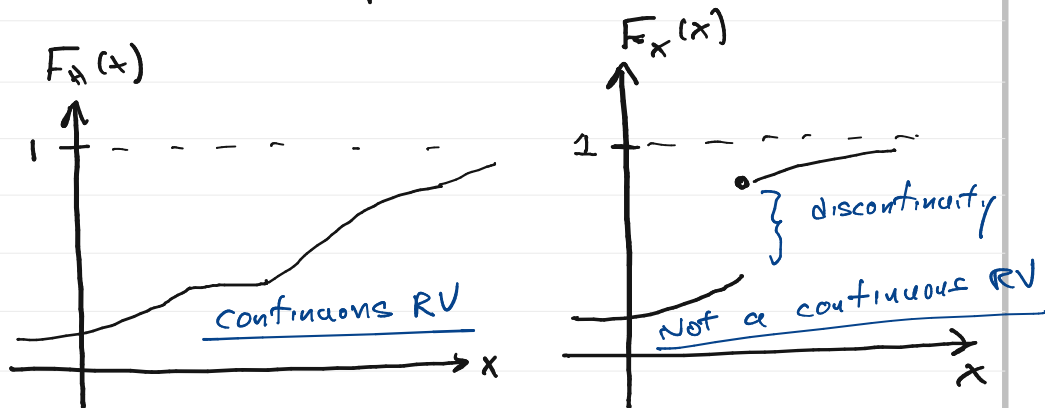
1. $F_X(+\infty) = 1$ and $F_X(-\infty) = 0$.
2. If $x_1 < x_2$, then $F_X(x_1) \leq F_X(x_2)$.
3. $P(\{X > x\}) = 1 - F_X(x)$, $\forall x \in \mathbb{R}$.
4. If $x_1 < x_2$, then
$$P(\{x_1 < X \leq x_2\}) = F_X(x_2) - F_X(x_1)$$
5. $P(\{X = x_0\}) = F_X(x_0) - F_X(x_0^-)$

where

$$F_X(x_0^-) = \lim_{\varepsilon \downarrow 0} F_X(x_0 - \varepsilon)$$

Defn: We say that a random variable is (absolutely) continuous if $F_X(x)$ is a continuous function at all points $x \in \mathbb{R}$.

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We say that a RV X is discrete if it takes on values from a discrete (finite or countable) subset of \mathbb{R} .

In this case, $F_X(x)$ is a "staircase function".

