

Session 10

Random Variables

10.1

"A function of the outcome of a random experiment taking on a numerical value."

Mappings and Functions:

Defn: Given two abstract spaces \mathcal{S} and \mathcal{A} , an \mathcal{A} -valued function mapping \mathcal{S} to \mathcal{A}

$$f: \mathcal{S} \rightarrow \mathcal{A}$$

is an assignment of a specific element of \mathcal{A} to each $w \in \mathcal{S}$.

Thus, given $w \in \mathcal{D}$, $f(w) \in \mathcal{A}$,
 $\forall w \in \mathcal{D}$.

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Given

$$f: \mathcal{D} \rightarrow \mathcal{A}$$

\mathcal{D} is called the domain of f ;

\mathcal{A} is called the range of f .

Example: In calculus we study real valued functions of a real variable:

$$f: \mathbb{R} \rightarrow \mathbb{R}.$$

$$\mathcal{D} = \mathbb{R} \text{ and } \mathcal{A} = \mathbb{R}.$$

Defn: Given any two sets $F \subset \mathcal{D}$
and $G \subset \mathcal{A}$, we define the
image of F under f as

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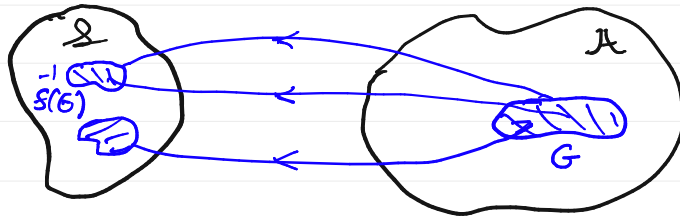
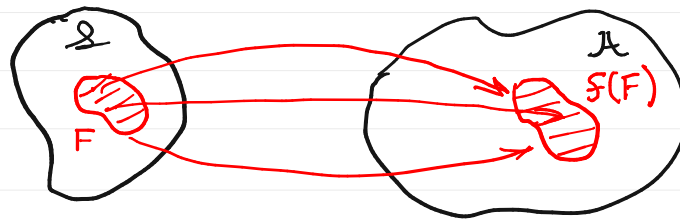
$$f(F) \triangleq \{a \in \mathcal{A} : a = f(w) \text{ for some } w \in F\}.$$

and the pre-image or inverse image
of $G \subset \mathcal{A}$ under f is defined as

$$f^{-1}(G) \triangleq \{w \in \mathcal{D} : f(w) \in G\}$$

Pictorially, the situation appears as follows:

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Random Variables

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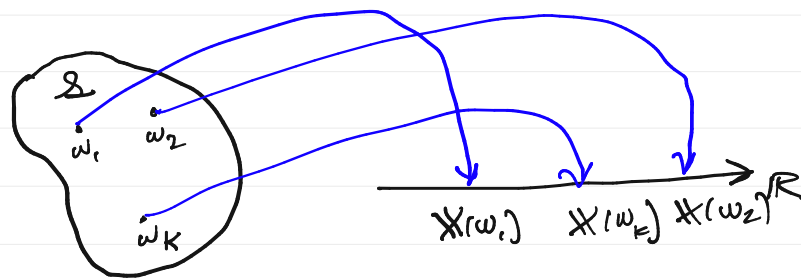
- We often characterize the outcome of a random experiment by a number (perhaps a measurement.)

e.g. - Induced electric current in an antenna due to random thermal motion of charges.

- The number of "Heads" that occur in N independent coin tosses.

Intuitive "Defn": Given $(\mathcal{S}, \mathcal{F}, P)$, 10.6
a random variable is a mapping
from \mathcal{S} to the real line.

$$X: \mathcal{S} \rightarrow \mathbb{R}$$



A random variable is a real-valued 10.7
function defined on \mathcal{S}
- \mathcal{S} is its domain
- \mathbb{R} is its range

1. It is not random.
2. It is not a variable.

The mapping $X: \mathcal{S} \rightarrow \mathbb{R}$ is fixed and deterministic
(non-random)

The randomness observed in $X(\omega)$
is due to the randomness of the outcome
 $\omega \in \mathcal{S}$ in the random experiment $(\mathcal{S}, \mathcal{F}, P)$

A function $X: \Omega \rightarrow \mathbb{R}$ has the property that its output inherits a probability measure from P in the underlying probability space (Ω, \mathcal{F}, P) :

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We require that $X(\cdot)$ has the property that for any $A \in \mathcal{B}(\mathbb{R})$, we can compute the probability $P_X(A)$.

Any function $X(\cdot)$ such that we can compute $P_X(A)$ for all $A \in \mathcal{B}(\mathbb{R})$ is called a (Borel) measurable function.

Defn: Given (Ω, \mathcal{F}, P) a random variable is a mapping $X: \Omega \rightarrow \mathbb{R}$ with the property that for all $A \in \mathcal{B}(\mathbb{R})$,

$$X^{-1}(A) = \{\omega \in \Omega : X(\omega) \in A\} \in \mathcal{F}$$

Such a function is called a Borel measurable function.

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(Borel) Measurability of X insures that we can compute the probability

10.10

that

$$\{X \in A\} = \{\omega \in \Omega : X(\omega) \in A\} \in \mathcal{F}$$

for all $A \in \mathcal{B}(\mathbb{R})$.

Q: How restrictive is measurability on the family of functions that are valid RVs?

e.g. If (Ω, \mathcal{F}, P) is taken to have

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$$\Omega = \mathbb{R}$$

$$\mathcal{F} = \mathcal{B}(\mathbb{R})$$

Then almost any function

$$X : \mathbb{R} \rightarrow \mathbb{R}$$

will be a random variable.

In fact, for $X(\cdot)$

$$(\mathbb{R}, \mathcal{B}(\mathbb{R}), P) \Rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}), P_X)$$

all of the following will be valid RVs:

1. Continuous functions

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2. Polynomials

3. Step functions: $k \cdot U_{-1}(x-a)$

4. All indicator functions:

$$\mathbb{1}_A(\omega) = \begin{cases} 1, & \omega \in A \\ 0, & \omega \notin A \end{cases}$$

for all $A \in \mathcal{B}(\mathbb{R})$.

5. All trig. functions (circular and hyperbolic)

6. All Bessel functions

7. Limits of sequences of measurable function

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$$f_1(x), f_2(x), \dots, f_n(x), \dots$$

$$\lim_{n \rightarrow \infty} f_n(x) = f(x)$$

8. Sums (finite and countable) of measurable functions

$$f_1(x), f_2(x), \dots, f_n(x), \dots:$$

$$f(x) = \sum_n f_n(x).$$

Fact: It is very difficult to describe a function $X(\cdot)$ that is not measurable.

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e.g. If A is one of the "problem sets" in $\mathcal{P}(\mathbb{R}) - \mathcal{B}(\mathbb{R})$, then

$$X(\omega) = \mathbb{1}_A(\omega)$$

is not measurable and hence not a random variable.

The "Range Space" of a function f

10.15

Let $f: \mathcal{S} \rightarrow A$.

We call the image $f(\mathcal{S}) \subset A$ the range space of f .

e.g. $f: \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = x^2$

then " $f(\mathbb{R})$ " = $[0, \infty)$ = range space of $f(x) = x^2$.

Given $(\mathcal{S}, \mathcal{F}, P)$ and a R.V. X , the R.V. takes on values in the range space

$$\mathcal{C} = f(\mathcal{S}) \subset \mathbb{R}.$$