Session 7

Recall...

Defn: Given $(\mathcal{S}, \mathcal{F}, P)$ and $A, B \in \mathcal{F}$, the conditional probability of A conditioned on B ("A given B") is

$$P(A|B) \triangleq \frac{P(A \cap B)}{P(B)},$$

assuming $P(B) \neq 0$.

Recall...

Fact: If $P(\cdot)$ (from $(\mathcal{S}, \mathcal{F}, P)$) is a valid probability measure, then $P(\cdot \mid B)$ is also a valid probability measure for any $B \in \mathcal{F}$ such that $P(B) \neq 0$.

7.2

Proof: (exercise) Verify the axioms of probability hold for $P(\cdot \mid B)$.

$$(\mathcal{S}, \mathcal{F}, P) \xRightarrow{\text{B has occurred}} (\mathcal{S}, \mathcal{F}, P(\cdot \mid B))$$

n.b. $(\mathcal{S}, \mathcal{F}, P(\cdot \mid B))$ is a valid prob. space because $(\mathcal{S}, \mathcal{F}, P)$ is a valid prob. space.

Bayes Formula, the Total Probability Law, and Bayes Theorem

Suppose I have a probability space $(\mathcal{S}, \mathcal{F}, P)$.
Let $A, B \in \mathcal{F}$

Then $$P(A|B) = \frac{P(A \cap B)}{P(B)} \quad \text{---} \quad (1)$$

and $$P(B|A) = \frac{P(A \cap B)}{P(A)} \quad \text{---} \quad (2)$$

From (2), we have

$$P(A \cap B) = P(B|A)\,P(A) \quad \text{---} \quad (2')$$

Substituting (2′) into (1), we get

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B|A)\,P(A)}{P(B)}$$

$$\therefore \quad \boxed{P(A|B) = \frac{P(B|A)\,P(A)}{P(B)}}$$

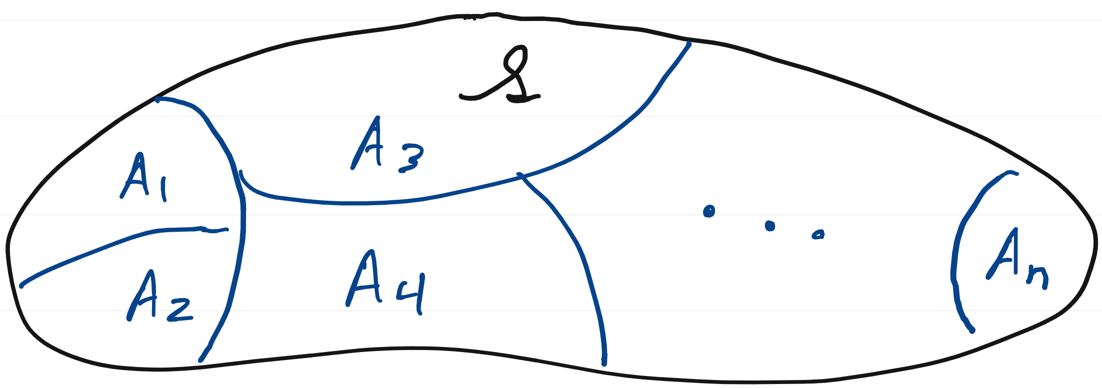
Bayes Formula.

The Total Probability Law

Given $(\mathcal{S}, \mathcal{F}, P)$, let $\{A_1, \ldots, A_n\}$ be a partition of $\mathcal{S}$, and let $B \in \mathcal{F}$.
(n.b., $A_1, A_2, \ldots, A_n \in \mathcal{F}$)

Then

$$P(B) = P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + \cdots + P(B|A_n)P(A_n)$$



Proof:

$$
\begin{aligned}
P(B) &= P(B \cap \mathcal{S}) \\
&= P\left(B \cap \left(\bigcup_{i=1}^{n} A_i\right)\right) \\
&= P\left(\bigcup_{i=1}^{n} (B \cap A_i)\right) \\
&= \sum_{i=1}^{n} P(B \cap A_i) \\
&= \sum_{i=1}^{n} P(B \mid A_i)\, P(A_i) \quad \blacksquare
\end{aligned}
$$

Bayes Theorem

Given $(\mathcal{S}, \mathcal{F}, P)$, assume that $\{A_1, \ldots, A_n\}$ is a partition of $\mathcal{S}$.

Then by Bayes' formula we have

$$P(A_i \mid B) = \frac{P(B \mid A_i)\, P(A_i)}{P(B)}$$

By the total prob. law, we have

$$P(B) = \sum_{i=1}^{n} P(B \mid A_i)\, P(A_i)$$

Suppose $A_m \in \{A_1, \dots, A_n\}$

By Bayes formula

$$P(A_m|B) = \frac{P(B|A_m)\,P(A_m)}{P(B)}$$

$$= \frac{P(B|A_m)\,P(A_m)}{\sum_{n=1}^{n} P(B|A_i)\,P(A_i)}$$

$$\therefore \boxed{P(A_m|B) = \frac{P(B|A_m)\,P(A_m)}{\sum_{i=1}^{n} P(B|A_i)\,P(A_i)}}$$

Bayes Theorem.

Bayes Theorem: Let $(\mathcal{S}, \mathcal{F}, P)$ be a probability space and $\{A_1, \dots, A_n\}$ be a partition of $\mathcal{S}$. Assume that $A_1, \dots, A_n \in \mathcal{F}$, and assume $B \in \mathcal{F}$. Then

$$P(A_m | B) = \frac{P(B|A_m)P(A_m)}{\sum_{i=1}^{n} P(B|A_i)\, P(A_i)}, \qquad m = 1, \dots, n.$$

Proof: We just proved it.

Statistical Independence

Defn: Given $(\mathcal{S}, \mathcal{F}, P)$, let $A, B \in \mathcal{F}$.

Then the events A and B are statistically independent if and only if (iff)

$$P(A \cap B) = P(A)\,P(B)$$

Fact: If A and B are statistically independent, then so are A and $\overline{B}$, $\overline{A}$ and B, and $\overline{A}$ and $\overline{B}$.

Proof for A and $\overline{B}$. We want to show

$$P(A \cap \overline{B}) = P(A)P(\overline{B})$$

given that $P(A \cap B) = P(A)P(B)$

$$A = \underbrace{(A \cap B)}_{} \cup \underbrace{(A \cap \overline{B})}_{}$$

disjoint

$$P(A) = P(A \cap B) + P(A \cap \overline{B})$$

$$\Rightarrow P(A \cap \overline{B}) = P(A) - P(A \cap B) = P(A) - P(A)P(B)$$

$$= P(A)[1 - P(B)] = P(A)P(\overline{B}) \quad \blacksquare$$

Statistical Independence:

3 events $A, B, C \in \mathcal{F}$ are statistically independent iff

$$P(A \cap B \cap C) = P(A)P(B)P(C)$$
$$P(A \cap B) = P(A)P(B)$$
$$P(A \cap C) = P(A)P(C)$$
$$P(B \cap C) = P(B)P(C).$$

In general, n events are statistically independent iff all possible combinations of intersections factor as

$$P(A_{j_1} \cap A_{j_2} \cap \cdots \cap A_{j_k}) = P(A_{j_1}) P(A_{j_2}) \cdots P(A_{j_k})$$

for all combinations of k events, where $k = 2, \ldots, n$.

There are $2^n - (n+1)$ such combinations to check.

There are $2^n - (n+1)$ such combinations:

$$\underbrace{\binom{n}{0} + \binom{n}{1}}_{n+1} + \underbrace{\binom{n}{2} + \binom{n}{3} + \cdots + \binom{n}{n}}_{x}$$

$$x + n + 1 = \binom{n}{0} 1^0 1^n + \binom{n}{1} 1^1 1^{n-1} + \cdots + \binom{n}{n} 1^n \cdot 1^0$$

$$= \sum_{k=0}^{n} \binom{n}{k} 1^k 1^{n-k} = (1+1)^n = 2^n$$

$$\Rightarrow \quad x = 2^n - (n+1) \quad \blacksquare$$

Combined Experiments

Suppose we have two random experiments $(\mathcal{S}_1, \mathcal{F}_1, \mathcal{P}_1)$ and $(\mathcal{S}_2, \mathcal{F}_2, \mathcal{P}_2)$.

We want to combine them to form a "super experiment" with probability space $(\mathcal{S}, \mathcal{F}, \mathcal{P})$, where

$$\mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2$$

<u>Example</u>: Exp. 1: flip a coin $\mathcal{S}_1 = \{H, T\}$

Exp. 2: Roll a die $\mathcal{S}_2 = \{1, 2, 3, 4, 5, 6\}$

The combined experiment has the sample space

$$\mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2$$

$$= \{(H,1), (H,2), (H,3), (H,4), (H,5), (H,6),$$
$$(T,1), (T,2) \quad \cdots \quad (T,6)\}.$$

<u>n.b</u> $|\mathcal{S}| = |\mathcal{S}_1| \cdot |\mathcal{S}_2|$.

An event in our new experiment will be a subset of the sample space

$$\mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2$$

If $A \subset \mathcal{S}_1$ and $B \subset \mathcal{S}_2$ $\left(\begin{array}{l} A \in \mathcal{F}_1 \\ B \in \mathcal{F}_2 \end{array} \right)$

Then

$$C = A \times B \subset \mathcal{S},$$

is an event in our new event space.

$$\left(\begin{array}{l} \underline{\text{e.g.}} \quad A = \{H\}, \quad B = \{3, 6\} \\ \qquad A \times B = \{(H,3), (H,6)\} \end{array} \right)$$

Our event space $\mathcal{F}$ will be the σ-field generated by all Cartesian products:

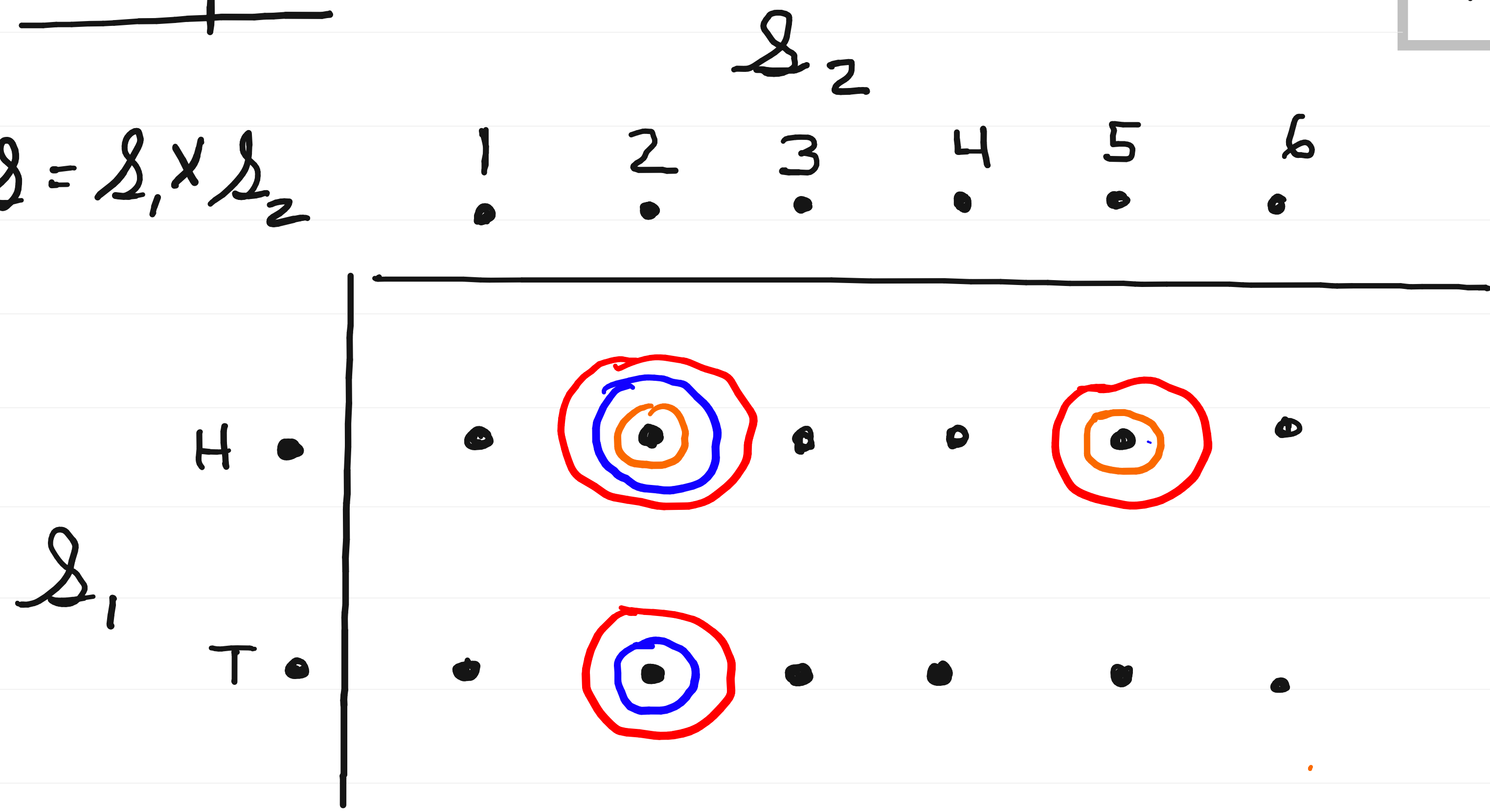
$$\mathcal{F} = \sigma\left(\{\underbrace{A \times B}_{\text{The cylinder sets}} : \forall A \in \mathcal{F}_1 \text{ and } \forall B \in \mathcal{F}_2\}\right)$$

This will be our event space in the combined experiment $(\mathcal{S}, \mathcal{F}, P)$

There are events that cannot be written as the cartesian product of events from $\mathcal{F}_1$ and $\mathcal{F}_2$, but the closure properties of the σ-field produce them:

Example

$S = S_1 \times S_2$



$C = \{H\} \times \{2, 5\} = \{(H,2), (H,5)\}$

$D = \{H, T\} \times \{2\} = \{(H,2), (T,2)\}$

$E = C \cup D = \{(T,2), (H,2), (H,5)\}$

How do we assign probabilities to the combined experiment $(\mathcal{S}, \mathcal{F}, P)$?

For consistency with P_1 and P_2, P of $(\mathcal{S}, \mathcal{F}, P)$ must satisfy

$$P(A \times \mathcal{S}_2) = P_1(A) \qquad \forall A \in \mathcal{F}_1$$

$$P(\mathcal{S}_1 \times B) = P_2(B), \quad \forall B \in \mathcal{F}_2$$

<u>Consistency conditions</u>

How do we assign other probabilities $P(C)$ for $C \in \mathcal{F}$?

How do we determine $P(C)$ for other events C ?

- We know that $P(C)$ must satisfy the consistency conditions.
- $P(C)$ must satisfy the axioms of probability.
- Other than that, we can't say much without further assumptions

Q: Is there a link or mechanism between the two constituent experiments?

Independent Experiments

Sometimes, the outcomes of the two constituent experiments are unrelated:

$$A \times \mathcal{S}_2 \perp\!\!\!\perp \mathcal{S}_1 \times B, \quad \forall A \in \mathcal{F}_1, \ \forall B \in \mathcal{F}_2$$

independent

In this case we say that the two experiments $(\mathcal{S}_1, \mathcal{F}_1, P_1)$ and $(\mathcal{S}_2, \mathcal{F}_2, P_2)$ are <u>independent experiments.</u>

For independent experiments, we assign the probability $P(\cdot)$ as

$$P(A \times B) = P\big((A \times \mathcal{S}_2) \cap (\mathcal{S}_1 \times B)\big)$$
$$= P(A \times \mathcal{S}_2) \cdot P(\mathcal{S}_1 \times B)$$
$$= P_1(A) \cdot P_2(B)$$

The axioms of probability fill in the probabilities of events that cannot be written as cartesian products (but can be written as a union of disjoint cartesian products.)