Session 5

Examples of Probability Spaces

Ex. 1: Let $\mathcal{S}$ be a finite sample space and let $\mathcal{F}(\mathcal{S})$ be the power set of $\mathcal{S}$.

Suppose we have a function $p(w) : \mathcal{S} \to \mathbb{R}$ such that

(i) $p(w) \geq 0, \ \forall w \in \mathcal{S}$

(ii) $\sum_{w \in \mathcal{S}} p(w) = 1.$

This function is called the <u>probability mass function</u> (pmf).

We can use the pmf to specify the prob. measure $\mathbb{P}(\cdot)$:

$$\mathbb{P}(A) = \sum_{\omega \in A} p(\omega), \quad \forall A \in \mathcal{F}.$$

This will be a valid prob. measure.

<u>n.b.</u> $p(\omega) = \mathbb{P}(\{\omega\})$, $\forall \omega \in \mathcal{S}$.

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Ex. 2: A uniform pmf

Let $S = \{\omega_1, \omega_2, \dots, \omega_n\}$ (finite S)

Let $\mathcal{F} = \mathcal{P}(S)$, $|\mathcal{P}(S)| = 2^n$

pmf: $p(\omega) = \frac{1}{n}$, $\forall \omega \in S$.

$$P(A_k) = \sum_{\omega \in A_k} p(\omega) = \sum_{\omega \in A_k} \left(\frac{1}{n}\right)$$

$$= \frac{1}{n} \sum_{\omega \in A_k} 1 = \frac{|A_k|}{n}$$

classical probability

$$= \frac{|A_k|}{|S|}, \quad \forall A_k \in \mathcal{F}$$

Ex. 3 Binomial pmf

$$\mathcal{S} = \{0, 1, 2, \ldots, n\}, \quad |\mathcal{S}| = n+1$$

$$\mathcal{F} = \mathcal{P}(\mathcal{S}), \quad |\mathcal{F}| = 2^{n+1}$$

pmf: $$p(k) = \binom{n}{k} a^k (1-a)^{n-k}, \quad a \in [0,1]$$

$$k = 0, 1, \ldots, n$$

where $$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

"n choose k"

$$P(A) = \sum_{k \in A} p(k), \quad \forall A \in \mathcal{F}.$$

Is this $p(k)$ a valid pmf?

(i) clearly, $p(k) = \underbrace{\binom{n}{k}}_{>0} \underbrace{a^k}_{\geq 0} \underbrace{(1-a)^{n-k}}_{\geq 0} \geq 0.$

(ii) Must show that $\sum_{k=0}^{n} \binom{n}{k} a^k (1-a)^{n-k} = 1$

* (exercise)

* Binomial Theorem:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

for any two numbers $a, b \in \mathbb{R}$ (or $a, b \in \mathbb{C}$).

You should memorize the Binomial Theorem.
We will use it many times in the course.

Ex. 4 The Geometric pmf:

$$\mathcal{S} = \{0, 1, 2, \ldots\} \quad (\mathcal{S} = \{1, 2, 3 \ldots\})$$

$$\mathcal{F} = \mathcal{P}(\mathcal{S})$$

pmf: $p(k) = (1-a)a^k, \quad a \in (0,1).$

$k = 0, 1, 2, \ldots$

$$P(A) = \sum_{k \in A} p(k) = \sum_{k \in A} (1-a)a^k, \quad \forall A \in \mathcal{F}.$$

Is this a valid pmf

(i) $p(k) = (1-a)a^k \geq 0, \quad k = 0, 1, 2, \ldots$

(ii) $P(\mathcal{S}) = \sum_{k=0}^{\infty} (1-a)a^k = 1^*$ (exercise)

$*$ Hint: $\sum_{k=0}^{\infty} a^k = \frac{1}{1-a}, \quad |a| < 1.$

Ex. 5: The Poisson pmf:

$$\mathcal{S} = \{0, 1, 2, \ldots\}$$
$$\mathcal{F} = \wp(\mathcal{S})$$

pmf: $$p(k) = \frac{\lambda^k e^{-\lambda}}{k!}, \quad k = 0, 1, 2, \ldots \quad \lambda > 0.$$

$$P(A) = \sum_{k \in A} p(k), \quad \forall A \in \mathcal{F}.$$

Is this a valid pdf?

(i) $p(k) = \dfrac{\lambda^k e^{-\lambda}}{k!} \geq 0, \quad k = 0, 1, 2, \ldots$

(ii) $P(\mathcal{S}) = \displaystyle\sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} = 1$ * (exercise)

* **Hint**: $$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

Let's look at an uncountable sample space:

<u>E.g.</u> $\mathcal{S} = \mathbb{R}$

$\mathcal{F} = \mathcal{B}(\mathbb{R})$

$P(\cdot)$ - How do we do this.

We introduce the <u>probability density function</u> (pdf) to assign probabilities $P(A)$, where $A \in \mathcal{B}(\mathbb{R})$.

Properties of the pdf:

A **probability density function** (pdf) is a function that maps the sample space $\mathcal{S} = \mathbb{R}$,

$$f : \mathcal{S} \to \mathbb{R}$$

satisfying the following properties:

(i) $f(r) \geq 0 \quad , \forall r \in \mathbb{R}$

(ii) $\displaystyle\int_{-\infty}^{\infty} f(r)\, dr = 1.$

Given a valid pdf $f(r)$, we get a valid probability measure $\mathbb{P}(\cdot)$ for any $A \in \mathcal{B}(\mathbb{R})$ by integrating:

$$\mathbb{P}(A) = \int_A f(r)\,dr = \int_{-\infty}^{\infty} f(r)\cdot \mathbb{1}_A(r)\,dr,$$

where

$$\mathbb{1}_A(r) = \begin{cases} 1, & r \in A \\ 0, & r \notin A \end{cases}$$

is called the <u>indicator function</u> of the set A

Q: Does $P(A) = \int_{-\infty}^{\infty} f(r)\cdot \mathbb{1}_A(r)\,dr$ give a valid prob. measure for every $A \in \mathcal{B}(\mathbb{R})$?

Consider the Riemann integral

$$\int_{-\infty}^{\infty} f(r)\cdot \mathbb{1}_A(r)\,dr$$

- It will be well defined for any A that is an interval.' $A = (a,b)$ or $[a,b]$, $a<b$.
- It's also well defined for an A equal to a <u>finite</u> union of intervals.

Example of a set A where the Riemann integral does not exist

Suppose I have a pdf

$$f(r) = 1_{[0,1]}(r) = \begin{cases} 1, & 0 \le r \le 1 \\ 0, & \text{elsewhere}. \end{cases}$$

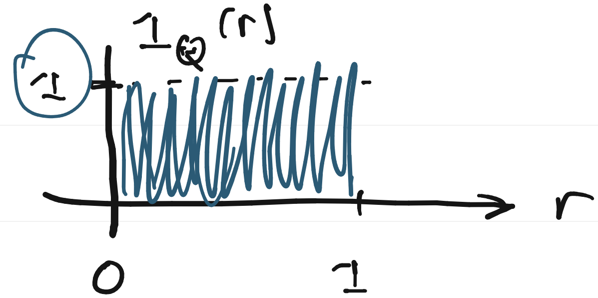
Let $A = \mathbb{Q}$ = rational numbers.

$$P(A) = P(\mathbb{Q}) = \int_{-\infty}^{\infty} f(r) \cdot 1_{\mathbb{Q}}(r)\, dr$$

$$= \int_0^1 1_{\mathbb{Q}}(r)\, dr$$

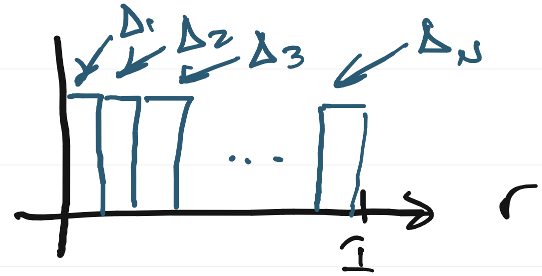
does not exist as a Riemann integral

$$P(\mathbb{Q}) = \int_{-\infty}^{\infty} f(r)\cdot 1_{\mathbb{Q}}(r)\,dr$$



$$= \int_0^1 1_{\mathbb{Q}}(r)\,dr$$

$$= \lim_{\substack{N\to\infty \\ \hat{x}_k \in \Delta_k}} \sum_{k=1}^{N} |\Delta_k|\cdot 1_{\mathbb{Q}}(\hat{x}_k)$$



$$\underline{\mathrm{Sum}}_N = \sum_{k=1}^{N} |\Delta_k|\cdot\left(\min_{x\in\Delta_k} 1_{\mathbb{Q}}(x)\right) = \sum_{k=1}^{N} |\Delta_k|\cdot 0 = 0$$

$$\overline{\mathrm{Sum}}_N = \sum_{k=1}^{N} |\Delta_k|\cdot\left(\max_{x\in\Delta_k} 1_{\mathbb{Q}}(x)\right) = \sum_{k=1}^{N} |\Delta_k|\cdot 1 = 1$$

For a Riemann integral to exist

$$\lim_{N\to\infty} \underline{\mathrm{Sum}}_N = \lim_{N\to\infty} \overline{\mathrm{Sum}}_N .$$

$\therefore$ The Riemann integral does not exist in this case.