Session 4

Recall...

Ex. 5 A space of countable sequences drawn from Ex. 1 - Ex. 3

$$\mathcal{S} = A \times A \times \cdots \times A \times \cdots$$

$$= \prod_{i \in \mathbb{N}} A = \prod_{i=1}^{\infty} A = A^{\mathbb{N}}$$

Examples: If I think of $A = \{H, T\}$ then $\mathcal{S} = \prod_{i=1}^{\infty} \{H, T\}$.

A typical element of $\mathcal{S}$ would be

$$(H, T, H, H, T, \ldots)$$

Even if A is a finite set, $\mathcal{S}$ will be uncountable

Why?

Because each sequence can be mapped to a point in $[0,1]$
(can be put into one-to-one correspondence.)

Let $\mathcal{S} = A^{\mathbb{N}}$, where $A = \{0,1\}$. Then a typical element in $\mathcal{S}$ would look like

$(a_1, a_2, \dots, a_n, \dots)$, $a_i \in \{0,1\}$

$$0.a_1 a_2 a_3 \dots = \sum_{j=1}^{\infty} \frac{a_j}{2^j} \in [0,1].$$

binary point

4.3

So $\mathcal{S} = A^{\mathbb{N}}$ can be put into one-to-one correspondence with $[0,1]$, which is uncountable.

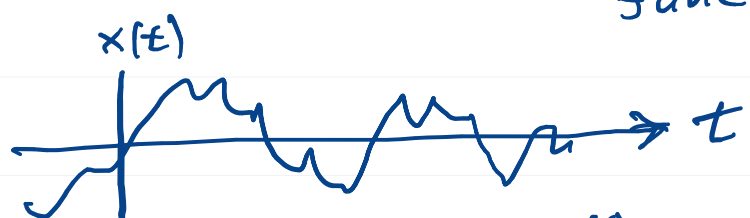
$\Rightarrow$ $\mathcal{S} = A^{\mathbb{N}}$ is uncountable

Ex.6: Let A be any sample space from Ex.1 to Ex.3

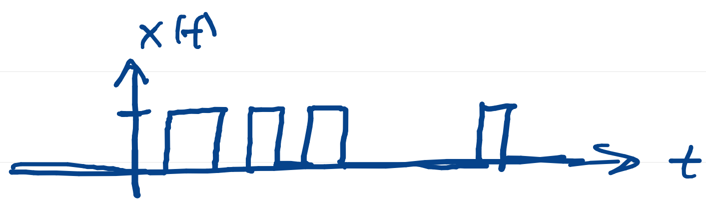
Let $$\mathcal{S} = \prod_{t \in \mathbb{R}} A$$

$$= \{\forall \text{ waveforms } x(t),\ t = (-\infty, +\infty),\ \text{with } x(t) \in A,\ \forall t \in (-\infty, +\infty)\}$$

e.g. $A = \mathbb{R} \Rightarrow \mathcal{S} =$ set of all real valued functions of t (time)



or if $A = \{0, 1\}$



Recall...

Event Spaces:

Intuitively: A collection of events (subsets of $\mathcal{S}$) that we are interested in computing the probability of.

Mathematically: $\mathcal{F}(\mathcal{S})$ or $\mathcal{F}$ is a family of subsets of $\mathcal{S}$ that satisfies certain closure properties

(σ - field)

Closure Properties: (exercise)

1. $A \in \mathcal{F}$, then $\overline{A} \in \mathcal{F}$.

2. If $A_1, A_2 \in \mathcal{F}$,
   Then $A_1 \cup A_2 \in \mathcal{F}$

3. If $A_1, A_2, \ldots, A_n, \ldots \in \mathcal{F}$,
   then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$.

Q: Why not construct probability theory using a field of sets (props. 1 and 2) instead of a σ-field (props. 1, 2, and 3)?

A: Probability Theory involves results expressed as limits of operations on sequences of events (Limit Theorems).

$\therefore$ we need countable sequences of set operations on sets to be in the event space.

Caution: What we have called a "σ-field", Papoulis calls a "Borel Field".

This is **not** correct!

Examples of Event Spaces

Ex. 1: Given any $\mathcal{S}$,

$\mathcal{F} = \{\phi, \mathcal{S}\}$ ("trivial event space")

is a valid event space

Ex. 2 Given any $\mathcal{S}$, the set of all subsets of $\mathcal{S}$ is a σ-field.

This set is called the power set of $\mathcal{S}$ and is denoted $\mathcal{P}(\mathcal{S})$ or $2^{\mathcal{S}}$.

- Both $\underline{\text{Ex. 1}}$ and $\underline{\text{Ex. 2}}$ are valid σ-fields for $\mathcal{S}$.

- Ex. 1: $\mathcal{F} = \{\phi, \mathcal{S}\}$ is not useful.
- Ex. 2: The power set $\mathcal{P}(\mathcal{S})$ is useful if $\mathcal{S}$ is finite or countable.

- However if $\mathcal{S}$ is uncountable (e.g., $\mathcal{S} = \mathbb{R}$ or $\mathcal{S} = [0,1]$) neither Ex. 1 or Ex. 2 is useful

  Ex. 1. Too small!
  Ex. 2: To big!

- If we take $\mathcal{S} = \mathbb{R}$ and $\mathcal{F} = \mathcal{P}(\mathbb{R})$, there are sets in $\mathcal{F} = \mathcal{P}(\mathbb{R})$ that we cannot assign probability to in such a way that satisfies the Axioms of Probability.

Let's construct a reasonable event space $\mathcal{F}$ for $\mathcal{S} = \mathbb{R}$.

Desired Properties of $\mathcal{F}(S)$ for $S = \mathbb{R}$.

1. We wish to include all events of the form

   $(a,b) = \{x \in \mathbb{R} : a < x < b\}$

   where $a, b \in \mathbb{R}$, $a < b$.

2. The closure properties of a σ-field are satisfied.

Defn: Given S and a family of subsets $G = \{A_i ; i \in I\}$ of S, the <u>σ-field generated by G</u>, denoted $\sigma(G)$, is the smallest σ-field containing all of the subsets in G.

n.b. By "smallest" σ-field, we mean that for any σ-field $\mathcal{F}_0$ containing the sets in G

$$\sigma(G) \subset \mathcal{F}_0$$

Back to $S = \mathbb{R}$

We want the smallest σ field containing all the open intervals

$$(a,b), \quad a<b, \quad a,b \in \mathbb{R}.$$

In our case, we want $\sigma(G)$

where

$$G = \{ (a,b), \ \forall a,b \in \mathbb{R} \ni a<b \}.$$

This σ-field $\sigma(G)$ of $\mathbb{R}$

$\sigma(\, G = \{ \text{all open intervals} \} \,)$

contains not only all open intervals, but also all countable sequences of set operations

$$\cup \;, \cap \;, \overline{}$$

on any collection of open intervals.

Defn: Given $\mathbb{R}$, the Borel field of $\mathbb{R}$ is defined as the σ-field generated by the family of all open intervals

$$G = \{(a,b) : \forall (a,b) \in \mathbb{R} \text{ such that } a < b\}.$$

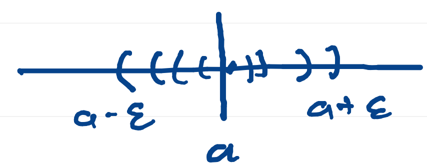
We denote the Borel field of $\mathbb{R}$ by $\mathcal{B}(\mathbb{R})$.

Because the Borel field contains all open intervals, it also contains:

$$(-\infty, b) = \bigcup_{n=1} (b-n, b) = \lim_{m\to\infty} (-m, b)$$

$$(a, +\infty) = \bigcup_{n=1}^{\infty} (a, a+n) = \lim_{m\to\infty} (a, m)$$

$$\{a\} = \bigcap_{n=1}^{\infty} \left(a-\frac{1}{n}, a+\frac{1}{n}\right)$$



$$= \lim_{n\to\infty} \left(a-\frac{1}{n}, a+\frac{1}{n}\right) = \{a\}$$

It also follows that

$$[a,b) = \{a\} \cup (a,b) \in \mathcal{B}(\mathbb{R})$$

$$(a,b] = (a,b) \cup \{b\} \in \mathcal{B}(\mathbb{R})$$

$$[a,b] = \{a\} \cup (a,b) \cup \{b\} \in \mathcal{B}(\mathbb{R})$$

In addition, all finite and countable sequences of set operations $\cup$ $\cap$, $\overline{}$ of these sets are in $\mathcal{B}(\mathbb{R})$.

" $\mathcal{B}(\mathbb{R})$ includes any subset of $\mathcal{S} = \mathbb{R}$ that we could be interested in. "

n.b. It is a difficult result in measure theory to show that

$$\mathcal{B}(\mathbb{R}) \neq \mathcal{P}(\mathbb{R}).$$

There are sets $A \subset \mathbb{R}$ that are not in $\mathcal{B}(\mathbb{R})$.
They are very strange and of no practical interest in probability problems.

n.b Sometimes we need to deal with a sample space

$$\mathcal{S} = A \subset \mathbb{R}. \quad (\text{e.g., } [0,1]).$$

In this case our event space should be the Borel field of A : $\mathcal{B}(A)$.

We can get the Borel field of $\mathcal{B}(A)$ by "cutting down" the Borel field of $\mathbb{R}$:

$$\mathcal{B}(A) = \{ F \cap A : \forall F \in \mathcal{B}(\mathbb{R}) \}$$

Probability Measures

Intuitively: Assigns a number between 0 and 1 that measures the certainty or "likelihood" that an event will occur.

Mathematically: A set function

$$P : \mathcal{F} \to \mathbb{R}$$

satisfying the Axioms of Probability.

Axioms of Probability

1. $P(A) \geq 0, \ \forall A \in \mathcal{F}$

2. $P(S) = 1$

3. If $A_1, A_2, \ldots, A_n \in \mathcal{F}$ and are disjoint, then

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i)$$

4. If $A_1, A_2, \ldots, A_n, \ldots \in \mathcal{F}$ and are disjoint

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

Other properties of $P(\cdot)$ that follow from the axioms:

1. $P(\phi) = 0$

2. $P(\bar{A}) = 1 - P(A)$

3. For any two events $A, B \in \mathcal{F}$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

<u>Proof</u>: Exercise.

Defn: A sequence of sets

$$A_1, A_2, \ldots, A_n, \ldots$$

is said to be <u>increasing</u> if

$$A_1 \subset A_2 \subset A_3 \subset \cdots \subset A_n \subset \cdots$$

and <u>decreasing</u> if

$$A_1 \supset A_2 \supset A_3 \supset \cdots \supset A_n \supset \cdots$$

Fact: If $A_1, A_2, \dots, A_n, \dots$ is an increasing sequence of sets, then

$$\lim_{n\to\infty} A_n = \bigcup_{i=1}^{\infty} A_i$$

$$\left(A_n = \bigcup_{i=1}^{n} A_i \right)$$

If $A_1, A_2, \dots, A_n, \dots$ is a decreasing sequence of sets, then

$$\lim_{n\to\infty} A_n = \bigcap_{i=1}^{\infty} A_i$$

$$\left(A_n = \bigcap_{i=1}^{n} A_i \right)$$

Fact: If $A_1, A_2, \dots, A_n, \dots$ is either an increasing sequence of sets or a decreasing sequence of sets, then

$$P\left(\lim_{n\to\infty} A_n\right) = \lim_{n\to\infty} P(A_n).$$

$\boxed{\text{Sequential continuity of the probability measure.}}$