Session 3

Recall...

The Sample Space $\mathcal{S}$

Defn: The sample space $\mathcal{S}$ is a non-empty set of possible outcomes of a random experiment.

One and only one outcome from the sample space occurs when we perform a random experiment.

Recall . . .

The Event Space $\mathcal{F}(\mathcal{S})$

3.2

Defn: The <u>event space</u> $\mathcal{F}(\mathcal{S})$ is a non-empty collection of subsets of $\mathcal{S}$ satisfying the following closure properties.

1. If $A \in \mathcal{F}(\mathcal{S})$, then $\bar{A} \in \mathcal{F}(\mathcal{S})$.
2. For any finite n, if $A_i \in \mathcal{F}(\mathcal{S})$ for $i = 1, 2, \ldots, n$, then

$$\bigcup_{i=1}^{n} A_i \in \mathcal{F}.$$

3. If $A_i \in \mathcal{F}(\Omega)$, $i = 1, 2, 3, \ldots$

then

$$\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}(\Omega).$$

A sets of subsets satisfying these 3 properties is called a <u>σ-field</u>.

(n.b. If only 1 and 2 hold, you have a <u>field</u> of sets.)

Recall...

Probability Measure

Defn: A probability measure $P(\cdot)$ (corresponding to $\mathcal{S}$ and $\mathcal{F}(\mathcal{S})$) is an assignment of a real number $P(A)$ to each $A \in \mathcal{F}(\mathcal{S})$ satisfying the Axioms of Probability

Axioms of Probability

1. $P(A) \geq 0, \quad \forall A \in \mathcal{F}(\mathcal{S})$.

2. $P(\mathcal{S}) = 1$.

3. If $A_1, A_2 \in \mathcal{F}(\mathcal{S})$ and $A_1 \cap A_2 = \phi$, then $P(A_1 \cup A_2) = P(A_1) + P(A_2)$

- If $\{A_1, \ldots, A_n\}$ (finite) are <u>disjoint</u> ($A_j \cap A_k = \phi$, $j \neq k$)

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i)$$

4. If $A_1, A_2, \ldots, A_n, \ldots \in \mathcal{F}(\mathcal{S})$ is a countable collection of disjoint events, then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i).$$

<u>n.b.</u> $P(\cdot)$ is a set function.

$P(\cdot) : \mathcal{F}(\mathcal{S}) \to \mathbb{R}$.

If we want to talk about the probability of a particular outcome $\omega_0 \in \mathcal{S}$, we do so by considering the singleton set $\{\omega_0\} \in \mathcal{F}(\mathcal{S})$

$P(\{\omega_0\})$ is well defined

$P(\omega_0)$ is <u>not</u> well defined

Examples of $(\mathcal{S}, \mathcal{F}, P)$:

Ex. 1 $\mathcal{S} = \{0, 1\}$

$\mathcal{F} = \{\phi, \mathcal{S}, \{0\}, \{1\}\}$

$$P(A) = \begin{cases} \quad , & A = \{0\} \\ 1-\alpha, & A = \{1\} \\ 0, & A = \phi \\ 1, & A = \mathcal{S} \end{cases}$$

where $0 \leq \alpha \leq 1$.

$P(A)$ satisfies the axioms of probability.

Ex. 2: Let $\mathcal{S}$ be any sample space.

Let $\mathcal{F}(\mathcal{S}) = \{\phi, \mathcal{S}\}$

By the axioms of prob., there is only one prob. measure $P(\cdot)$ we can assign:

$$P(A) = \begin{cases} 1, & A = \mathcal{S} \\ 0, & A = \phi \end{cases}$$

Empty sets: Good notation: ϕ and $\{\ \}$

Bad notation: $\cancel{\{\phi\}}$

It follows from the axioms of probability that

$$P(\phi) = 0.$$

<u>Proof</u>: $\mathcal{S} = \mathcal{S} \cup \phi$

disjoint ~ $\mathcal{S} \cap \phi = \phi$

$$\Rightarrow \quad P(\mathcal{S}) = P(\mathcal{S} \cup \phi) = \underbrace{P(\mathcal{S})}_{1} + P(\phi)$$

$$1 = 1 + P(\phi)$$

$$\Rightarrow \quad P(\phi) = 1 - 1 = 0 \; \blacksquare$$

Similarly, it follows from the axioms of probability that

$$P(\bar{A}) = 1 - P(A).$$

Proof:

$$\mathcal{S} = \underbrace{A \cup \bar{A}}_{\text{disjoint}}$$

$$1 \overset{\text{ax.2}}{=} P(\mathcal{S}) = P(A \cup \bar{A}) \overset{\text{ax.3}}{=} P(A) + P(\bar{A})$$

$$\Rightarrow \quad P(A) + P(\bar{A}) = 1$$

$$\Rightarrow \quad P(\bar{A}) = 1 - P(A). \blacksquare$$

Brief Summary:

- Random Experiment $\Rightarrow$ random outcome
- $\mathcal{S}$ is the set of all possible outcomes.
- Events are described as subsets of $\mathcal{S}$.
- If $A \subset \mathcal{S}$, we say that event A has occured if the random outcome $\omega \in \mathcal{S}$ is in A.
- Events of interest are collected into the event space $\mathcal{F}(\mathcal{S})$. $\mathcal{F}(\mathcal{S})$ also includes subsets generated by the closure properties. (σ-field).

- The probability that an event $A \in \mathcal{F}(\mathcal{S})$ occurs is given by $P(A)$

$$P(\cdot) : \mathcal{F}(\mathcal{S}) \longrightarrow \mathbb{R}$$

and satisfies the axioms of probability

We now take a more detailed look at $\mathcal{S}$, $\mathcal{F}$, and P:

Sample Space, $\mathcal{S}$:

Intuitively: Set containing all possible outcomes of a random experiment.

Mathematically: Universal Set (set of all outcomes.)

Ex. 1: A finite sample space:

$$\mathcal{S} = \{ w_k : k = 1, 2, \ldots, n \}$$

specifically: $\mathcal{S} = \{0, 1\}$, $\mathcal{S} = \{H, T\}$

$$\mathcal{S} = \{1, 2, 3, 4, 5, 6\}$$

Ex. 2: A countable sample space:

$$\mathcal{S} = \{ w_k : k = 1, 2, 3, \ldots \}$$

specifically: $\mathbb{N} = \{1, 2, 3, \ldots\}$

$$\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$$

$$\mathbb{Z} = \{ \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots \}$$

$\mathbb{Z}$ is countable because it can be put into one-to-one correspondence with $\mathbb{N}$:

$$
\begin{array}{lccccccccc}
\mathbb{Z}: & \ldots, & -3, & -2, & -1, & 0, & 1, & 2, & 3, & \ldots \\
 & & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\
\mathbb{N}: & \ldots & 7 & 5 & 3 & 1 & 2 & 4 & 6 & \ldots
\end{array}
$$

The Rational Numbers are Countable

Denominator, n

Numerator, m

	1	2	3	4	5	6	7	8	...
1	$\frac{1}{1}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$	$\frac{1}{8}$	...
2	$\frac{2}{1}$	$\frac{2}{2}$	$\frac{2}{3}$	$\frac{2}{4}$	$\frac{2}{5}$	$\frac{2}{6}$	$\frac{2}{7}$	$\frac{2}{8}$	...
3	$\frac{3}{1}$	$\frac{3}{2}$	$\frac{3}{3}$	$\frac{3}{4}$	$\frac{3}{5}$	$\frac{3}{6}$	$\frac{3}{7}$	$\frac{3}{8}$	...
4	$\frac{4}{1}$	$\frac{4}{2}$	$\frac{4}{3}$	$\frac{4}{4}$	$\frac{4}{5}$	$\frac{4}{6}$	$\frac{4}{7}$	$\frac{4}{8}$	...
5	$\frac{5}{1}$	$\frac{5}{2}$	$\frac{5}{3}$	$\frac{5}{4}$	$\frac{5}{5}$	$\frac{5}{6}$	$\frac{5}{7}$	$\frac{5}{8}$	...
6	$\frac{6}{1}$	$\frac{6}{2}$	$\frac{6}{3}$	$\frac{6}{4}$	$\frac{6}{5}$	$\frac{6}{6}$	$\frac{6}{7}$	$\frac{6}{8}$	...
7	$\frac{7}{1}$	$\frac{7}{2}$	$\frac{7}{3}$	$\frac{7}{4}$	$\frac{7}{5}$	$\frac{7}{6}$	$\frac{7}{7}$	$\frac{7}{8}$	...
8	$\frac{8}{1}$	$\frac{8}{2}$	$\frac{8}{3}$	$\frac{8}{4}$	$\frac{8}{5}$	$\frac{8}{6}$	$\frac{8}{7}$	$\frac{8}{8}$	...
⋮	⋮								

Consider $\frac{31}{255}$. It sits in the array at the intersection of row 31 and column 255.

Ex. 3 $S = (\alpha, \beta)$, $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$.

$= \{x \in \mathbb{R} : \alpha < x < \beta\}$

This is an uncountable set.

$[\alpha, \beta] = \{x \in \mathbb{R} : \alpha \leq x \leq \beta\}$
is uncountable.

$\mathbb{R} = (-\infty, +\infty)$ is uncountable.

* Let $\alpha = 0$ and $\beta = 1$

$\Rightarrow [0, 1]$ is uncountable.

Ex. 4 Let $\mathcal{S}$ be the set of all k-dim vectors whose coordinate (elements) come from any of the examples Ex. 1 – Ex. 3

(Call the "$\mathcal{S}$" of the previous example A)

$$\mathcal{S} = \underbrace{A \times A \times \cdots \times A}_{\substack{k\text{-fold} \\ \text{Cartesian product}}} = \prod_{i=1}^{k} A$$

Examples:
- State of a control system ($A = \mathbb{R}$).
- Length k binary codeword ($A = \{0, 1\}$)
- Length k discrete-time signal ($\mathcal{S} = \mathbb{R}$ or $\mathbb{C}$).

How can we see that the set $[\alpha, \beta]$ is uncountable?

Lets look at $[0,1]$. Prove this is uncountable.

Assume $[0,1]$ is countable. Then we can list the elements of $[0,1]$ in one-to-one correspondence with the natural numbers.

$$\begin{array}{ccl} \mathbb{N} & & \\ 1 & \leftrightarrow & 0.\,d_1^{(1)}\ d_2^{(1)}\ d_3^{(1)} \cdots \\ 2 & \leftrightarrow & 0.\,d_1^{(2)}\ d_2^{(2)}\ d_3^{(2)} \cdots \\ 3 & \leftrightarrow & 0.\,d_1^{(3)}\ d_2^{(3)}\ d_3^{(3)} \cdots \\ \vdots & & \quad\ \vdots \qquad \vdots \qquad \vdots \end{array}$$

$d_j^{(k)} = \{0,1\}$

the value of the k-th number is

$$\sum_{j=1}^{\infty} \frac{d_j^{(k)}}{2^j}$$

Consider the diagonal $d_j^{(j)}$ for $j = 1, 2, 3, \ldots$

What about the number

$$0.\tilde{d}_1\tilde{d}_2\tilde{d}_3\cdots$$

where $\tilde{d}_k \neq d_k^{(k)}$

$\tilde{d}_k \in \{0,1\}$

Cantor Diagonalization Method

So the number

$$\underline{0.\tilde{d}_1\tilde{d}_2\tilde{d}_3\cdots} = \sum_{j=1}^{\infty}\frac{\tilde{d}_j}{\underline{2^j}} \in [0,1]$$

But it's not in the original countable list

$\Rightarrow \Leftarrow$ Contradiction $\Rightarrow$ $[0,1]$ is not countable

i.e., $[0,1]$ is uncountable

Ex. 5 A space of countable sequences drawn from Ex. 1 – Ex. 3

$$\mathcal{S} = A \times A \times \cdots \times A \times \cdots$$

$$= \prod_{i \in \mathbb{N}} A = \prod_{i=1}^{\infty} A = A^{\mathbb{N}}$$