Session 29

Recall...

29.1

Example: Consider a R.P.

$$\mathbb{X}(t) = \cos(\omega_0 t + \Theta)$$

where Θ is a R.V. uniformly distributed on $[0, 2\pi)$:

$$f_{\Theta}(\theta) = \begin{cases} \frac{1}{2\pi}, & 0 \le \theta < 2\pi \\ 0, & \text{elsewhere} \end{cases}, \quad \omega_0 = \text{constant}$$

(radian frequency, not outcome)

Is $\mathbb{X}(t)$ W.S.S? Let's check the two defining conditions

Recall...

29.2

(i) $E[X(t)] = E[\cos(\omega_0 t + \Theta)]$

$$= E[\cos(\omega_0 t)\cos(\Theta) - \sin(\omega_0 t)\sin(\Theta)]$$

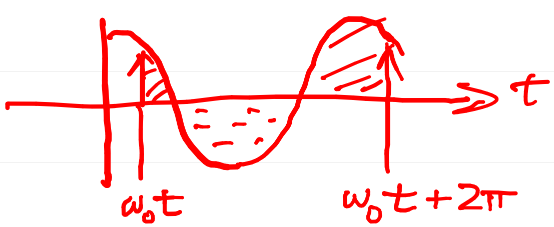
$$= \int_{-\infty}^{\infty} \left(\cos(\omega_0 t)\cos\theta - \sin(\omega_0 t)\sin(\theta)\right) f_{\Theta}(\theta)\, d\theta$$

$$= \frac{\cos \omega_0 t}{2\pi} \cancelto{0}{\int_0^{2\pi} \cos\theta\, d\theta} - \frac{\sin \omega_0 t}{2\pi} \cancelto{0}{\int_0^{2\pi} \sin\theta\, d\theta}$$

$= 0 \quad \therefore \quad E[X(t)] = 0 = \text{constant}.$ ☑

n.b. Even easier:

$$E[X(t)] = E[\cos(\omega_0 t + \Theta)] = \frac{1}{2\pi}\underbrace{\cancelto{0}{\int_0^{2\pi} \cos(\omega_0 t + \theta)\, d\theta}}_{\text{Integral over one period is 0.}}$$



Recall...

$$\cos A \cdot \cos B = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$$

29.3

(ii) : $E[X(t_1)X(t_2)]$

$$= E\left[\cos(w_0 t_1 + \Theta)\cdot\cos(w_0 t_2 + \Theta)\right]$$

$$= E\left[\frac{1}{2}\left(\cos(w_0(t_1+t_2)+2\Theta) + \cos(w_0(t_1-t_2))\right)\right]$$

$$= \frac{1}{2}E\left[\cos(w_0(t_1+t_2)+2\Theta)\right] + \frac{1}{2}E\left[\cos(w_0(t_1-t_2))\right]$$

0

$$= \frac{1}{2}\cos(w_0(t_1-t_2))$$ ☑

Recall...

$\therefore$ (i) $E[X(t)] = 0$ = constant,

(ii) $E[X(t_1)X(t_2)] = \frac{1}{2}\cos w_0(t_1 - t_2)$.

$\therefore$ $X(t)$ is a W.S.S. R.P.

Another Example:

Let $\Psi(t) = \cos(\omega_0 t + \phi)$

where $\omega_0 = \text{constant}$

and ϕ is a uniform R.V. on $[0, \pi)$.

Is $\Psi(t)$ W.S.S.?

(i) $E[\Psi(t)] = E[\cos(\omega_0 t + \phi)]$

$$= \frac{1}{\pi}\int_0^{\pi} \cos(\omega_0 t + \phi)\, d\phi$$

$$= \frac{1}{\pi}\int_0^{\pi} [\cos\omega_0 t \cos\phi - \sin\omega_0 t \sin\phi]\, d\phi$$

$$= \frac{\cos \omega_0 t}{\pi} \int_0^{\pi} \cancel{\cos \phi}^{\,0} \, d\phi - \frac{\sin \omega_0 t}{\pi} \int_0^{\pi} \sin \phi \, d\phi$$

$$= \frac{\sin \omega_0 t}{\pi} \Big[-\cos \phi \Big]_0^{\pi} = -\frac{2}{\pi} \sin \omega_0 t$$

$$\neq \text{constant}$$

$\therefore E[Y(t)] \neq$ constant

$\Rightarrow Y(t)$ is not W.S.S.

Defn: The mean of a R.P. $X(t)$ is

$$\mu_X(t) \triangleq E[X(t)]$$

Defn: The autocorrelation function of a R.P. $X(t)$ is

$$R_{XX}(t_1, t_2) \triangleq E[X(t_1)X(t_2)]$$

n.b. The autocorrelation function is just a measure of the correlation in $X(t)$ at time t_1 and time t_2.

n.b. $R_{XX}(t_1,t_2) = E[\mathbf{X}(t_1)\mathbf{X}(t_2)]$

$$= E[\mathbf{X}(t_2)\mathbf{X}(t_1)] = R_{XX}(t_2,t_1)$$

for a real andom process.

n.b The autocorrelation function is symmetric in its time arguments.

For a complex random process $\mathbf{X}(t)$

$$R_{XX}(t_1,t_2) = E[\mathbf{X}(t_1)\mathbf{X}^*(t_2)] = \left(E[\mathbf{X}(t_2)\mathbf{X}(t)^*]\right)^*$$

$$= R_{XX}^*(t_2,t_1)$$

Defn: The **autocovariance function** of a real random process $X(t)$ is defined as

$$C_{XX}(t_1,t_2) \triangleq E\left[\left(X(t_1)-\mu_X(t_1)\right)\left(X(t_2)-\mu_X(t_2)\right)\right]$$

$$\Big(\underbrace{= \cdots = R_{XX}(t_1,t_2)-\mu_X(t_1)\,\mu_X(t_2)}_{\text{can easily show this}} \Big)$$

Defn: A random process $X(t)$ is called a <u>Gaussian random process</u> if the RVs

$X(t_1), X(t_2), \ldots, X(t_n)$

are jointly Gaussian for any $n \in \mathbb{N}$ and any set of sample times $t_1, \ldots, t_n$.

Fact: The n-th order characteristic function of a Gaussian R.P. $\mathbb{X}(t)$ is

$$\Phi_{\mathbb{X}(t_1),\ldots,\mathbb{X}(t_n)}(\omega_1,\ldots,\omega_n) = \exp\left\{i\sum_{j=1}^{n}\mu_X(t_j)\,\omega_j\right\}\exp\left\{-\frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}C_{XX}(t_j,t_k)\,\omega_j\omega_k\right\}$$

This is a complete probabilistic description.

Fact: A Gaussian random process $\mathbb{X}(t)$ is completely characterized by

(i) $\mu_X(t) = E[\mathbb{X}(t)]$

and

(ii) $C_{XX}(t_1,t_2) = E\left[(\mathbb{X}(t_1)-\mu_X(t_1))(\mathbb{X}(t_2)-\mu_X(t_2))\right]$

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Note: If a R.P $X(t)$ is W.S.S., then

$$R_{XX}(t_1,t_2) = E[X(t_1)X(t_2)]$$

$$= f(t_1-t_2)$$

$$= \text{"} R_X(t_1-t_2) \text{"}$$

This is also sometimes written as

$$E[X(t+\tau)X(t)] = R_X(\tau)$$

for a W.S.S. random process.

Defn: If $X(t)$ is a w.s.s. random process with autocorrelation function $R_X(\tau)$, then the

Power Spectral Density

of $X(t)$ is defined as

$$S_{XX}(\omega) \triangleq \int_{-\infty}^{\infty} R_X(\tau)\, e^{-i\omega\tau}\, d\tau$$

Here ω is radian frequency (not the outcome of experiment.)

$S_{XX}(\omega)$ is a measure of the average distribution of signal power in frequency for the R.P. $X(t)$.

Systems with Stochastic Inputs

$$\mathbb{X}(t) \longrightarrow \boxed{\;T\;} \longrightarrow \mathbb{Y}(t)$$

$\mathbb{X}(t,\omega)$ $\qquad\qquad\qquad\qquad$ $\mathbb{Y}(t,\omega)$

Given a R.P. $\mathbb{X}(t)$, if we assign to each sample function $\mathbb{X}(t,\omega)$ a new sample function $\mathbb{Y}(t,\omega)$, we have a new random process

$$\mathbb{Y}(t) = T[\mathbb{X}(t)]$$

whose sample functions are

$$\mathbb{Y}(t,\omega) = T[\mathbb{X}(t,\omega)].$$

<u>n.b.</u> We will assume that $T[\cdot]$ is <u>deterministic</u>. (not random)

Think of

$X(t)$ = input to a system

$Y(t)$ = output of the system

$$X(t) \longrightarrow \boxed{T} \longrightarrow Y(t)$$

$$X(t,\omega) \longrightarrow \boxed{T} \longrightarrow Y(t,\omega)$$

$$Y(t,\omega) = T[X(t,\omega)], \; \forall \omega \in \Omega.$$

- We are interested in finding a statistical description of the output $Y(t)$ in terms of the statistical description of the input $X(t)$ and the system description $T[\cdot]$

- For general T this is very difficult
- We will look at two special cases:
  1. Linear Time Invariant (L.T.I.) Systems
  2. Memoryless Systems.

Linear Systems

A linear system $\overset{T}{\downarrow}$ $L[\cdot]$ is a transformation rule satisfying the following two properties:

1. $L[\mathbb{X}_1(t) + \mathbb{X}_2(t)] = L[\mathbb{X}_1(t)] + L[\mathbb{X}_2(t)]$

(Superposition)

2. $L[\mathbb{A} \cdot \mathbb{X}(t)] = \mathbb{A} \cdot L[\mathbb{X}(t)]$

(Homogeneity)

n.b. $\mathbb{A}$ can be an R.V. or a constant

Defn: A (linear) system is <u>time-invariant</u> if, given response $y(t)$ to input $x(t)$ it has response $y(t+c)$ for input $x(t+c)$, for all $c \in \mathbb{R}$.

A linear time invariant system is characterized by its impulse response $h(t)$:

$$\delta(t) \longrightarrow \boxed{h(t)} \longrightarrow h(t)$$

$$\delta(t-t_0) \qquad\qquad\qquad h(t-t_0)$$

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = x(t) * h(t).$$

If we put a random process $\mathbb{X}(t)$ into a L.T.I. system, we get a random process $\mathbb{Y}(t)$ out of the system:

$$\mathbb{Y}(t) = \mathbb{X}(t) * h(t) = \int_{-\infty}^{\infty} \mathbb{X}(t-\alpha)\, h(\alpha)\, d\alpha$$

$$= \int_{-\infty}^{\infty} \mathbb{X}(\alpha)\, h(t-\alpha)\, d\alpha$$

We interpret this on a sample function basis:

$$\mathbb{Y}(t,\omega) = \mathbb{X}(t,\omega) * h(t), \quad \forall \omega \in \Omega.$$

Important Facts:

1. If the input to a L.T.I. system is a Gaussian R.P., then the output is a Gaussian R.P.

2. If the input to a stable L.T.I. system is S.S.S., so is the output.

An L.T.I. system is stable iif (BIBO stable)

$$\int_{-\infty}^{\infty} |h(t)| \, dt < \infty .$$

Fundamental Theorem:

For any linear system

$$E[L[\mathbb{X}(t)]] = L[E[\mathbb{X}(t)]]$$

(This basically reduces to an exchange of orders of integration).

Applying this to a L.T.I. system, we get

$$E[Y(t)] = E\left[\int_{-\infty}^{\infty} X(t-\alpha)\, h(\alpha)\, d\alpha\right]$$

$$= \int_{-\infty}^{\infty} E[X(t-\alpha)]\, h(\alpha)\, d\alpha$$

$$= \int_{-\infty}^{\infty} \mu_X(t-\alpha)\, h(\alpha)\, d\alpha$$

$$= \mu_X(t) * h(t)$$

$$\therefore \mu_Y(t) = E[Y(t)] = \mu_X(t) * h(t).$$

$$R_{YY}(t_1,t_2) = E[\,Y(t_1)\,Y(t_2)\,]$$

$$= E\Big[\underbrace{\int_{-\infty}^{\infty} X(t_1-\alpha)\,h(\alpha)\,d\alpha}_{Y(t_1)} \cdot \underbrace{\int_{-\infty}^{\infty} X(t_2-\beta)\,h(\beta)\,d\beta}_{Y(t_2)}\Big]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E[\,X(t_1-\alpha)\,X(t_2-\beta)\,]\,h(\alpha)\,h(\beta)\,d\alpha\,d\beta$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{XX}(t_1-\alpha,\,t_2-\beta)\,h(\alpha)\,h(\beta)\,d\alpha\,d\beta$$

If $R_{XX}(t_1,t_2) = R_X(t_1-t_2)$, then we would have

$$R_{YY}(t_1,t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_X(\underbrace{t_1-t_2}_{}-\alpha+\beta)\,h(\alpha)\,h(\beta)\,d\alpha\,d\beta$$

$$= R_Y(t_1-t_2) \quad \text{(a function of } t_1-t_2\text{)}$$

So if $X(t)$ is a W.S.S. random process, then

(i) $\mu_Y(t) = \mu_X * h(t) = \int_{-\infty}^{\infty} \mu_X h(t-\alpha)\, d\alpha$

$$= \mu_X \int_{-}^{\infty} h(t-\alpha)\, d\alpha = \mu_Y$$

(ii) $R_{YY}(t_1, t_2) = R_Y(t_1 - t_2)$

<u>Theorem</u>: If the input to a stable L.T.I. system is a W.S.S. random process, then the output is a W.S.S. random process.

Recall the definition of the Power Spectral Density of a W.S.S R.P. $X(t)$ is

ω = radian freq.

$$S_{XX}(\omega) \triangleq \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-i\omega\tau} \, d\tau$$

$\tau = t_1 - t_2$

(where we assume $X(t)$ is W.S.S.)

where

$$R_{XX}(\tau) = E[X(\underbrace{t+\tau}_{t_1}) X^*(\underbrace{t}_{t_2})]$$

for a W.S.S. R.P. $X(t)$

Note: 1. Because $R_{xx}(-\tau) = R_{xx}^{*}(\tau)$

$S_{xx}(\omega)$ is a real function

2. $$R_{xx}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)\, e^{i\omega\tau}\, d\omega$$

3. In order to consider $S_{xx}(\omega)$, we must assume $X(t)$ is W.S.S.

4. Because $R_{xx}(\tau)$ is a non-negative definite function, it follows that $S_{xx}(\omega) \geq 0$.

Key Result: If $X(t)$ is a W.S.S. Random Process and it is the input to a stable L.T.I. system with impulse response $h(t)$, then the Power Spectral Density of the output $Y(t)$ is

$$\boxed{S_{YY}(\omega) = S_{XX}(\omega)\,|H(\omega)|^2}$$

where

$$H(\omega) = \int_{-\infty}^{\infty} h(t)e^{-i\omega t}\,dt.$$

Defn: A random process $\mathbb{W}(t)$ is called a white noise process if

$$C_{\mathbb{W}\mathbb{W}}(t_1, t_2) = 0, \quad \forall t_1 \neq t_2.$$

Fact: All (non-trivial) W.S.S. white noise processes have

$$R_{WW}(t_1, t_2) = r_0 \cdot \delta(t_1 - t_2)$$

where $r_0 = \text{constant} > 0$.

EXAMPLE: Let $X(t)$ be a W.S.S R.P. with PSD $S_{XX}(\omega)$ and let $Y(t)$ be the "smoothed" random process given by

$$Y(t) = \frac{1}{2T} \int_{t-T}^{t+T} X(\alpha)\, d\alpha$$

This can be represented by a L.T.I. System with impulse response

$$h(t) = \frac{1}{2T} \cdot \mathbb{1}_{[-T,T]}(t).$$

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$$X(t) \longrightarrow \boxed{h(t)} \longrightarrow Y(t)$$

$$h(t) = \frac{1}{2T} \cdot 1_{[-T,T]}(t)$$

What is the PSD $S_{YY}(\omega)$ of $Y(t)$?

Solution:

$$\boxed{S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega)}$$

$$H(\omega) = \int_{-\infty}^{\infty} h(t) e^{-i\omega t} \, dt = \int_{-\infty}^{\infty} \frac{1}{2T} 1_{[-T,T]}(t) \, e^{-i\omega t} \, dt$$

$$= \frac{1}{2T} \int_{-T}^{T} e^{-i\omega t} \, dt = \frac{1}{2T} \left. \frac{e^{-i\omega t}}{-i\omega} \right|_{-T}^{T}$$

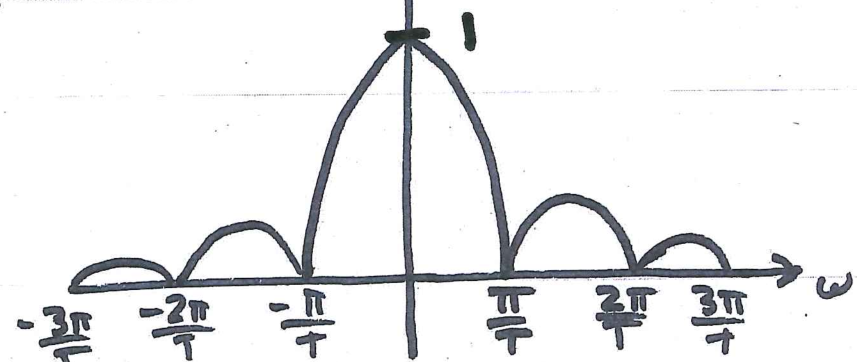
$$= \frac{e^{-i\omega T} - e^{+i\omega T}}{-i2T\omega} = \frac{1}{T\omega} \left[\frac{e^{i\omega T} - e^{-i\omega T}}{i2} \right] = \frac{\sin(T\omega)}{T\omega}$$

$\therefore$

$$\boxed{S_{YY}(\omega) = S_{XX}(\omega)\left|\frac{\sin(T\omega)}{T\omega}\right|^2}$$

Note that $h(t)$ acts as a crude low-pass filter that attenuates high-freq. power in the signal.

$$|H(\omega)|^2 = \left|\frac{\sin(T\omega)}{T\omega}\right|^2$$



Given a W.S.S RP $X(t)$ that is the input to an L.T.I system:

$$X(t) \longrightarrow \boxed{h(t)} \longrightarrow Y(t)$$

If I want to find the autocorrelation function of the output I have two options:

$$(1) \quad R_Y(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_X(\tau-\alpha+\beta)\,h(\alpha)\,h(\beta)\,d\alpha\,d\beta$$

Most of the time this is a complicated calculation.

(2) Use $S_{YY}(\omega) = |H(\omega)|^2 \cdot S_{xx}(\omega)$

(a) Compute

$$S_{xx}(\omega) = \int_{-\infty}^{\infty} R_x(\tau) e^{-i\omega\tau} dt$$

(b) Compute

$$H(\omega) = \int_{-\infty}^{\infty} h(t) e^{-i\omega t} dt$$

(c) Compute

$$S_{YY}(\omega) = S_{xx}(\omega) \cdot |H(\omega)|^2$$

(d) Compute

$$R_Y(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{YY}(\omega) e^{+i\omega\tau} d\omega$$

Often the inverse Fourier transform can be looked up in a table:

TABLE 10-1*

$$R(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)e^{j\omega\tau}\,d\omega \leftrightarrow S(\omega) = \int_{-\infty}^{\infty} R(\tau)e^{-j\omega\tau}\,d\tau$$

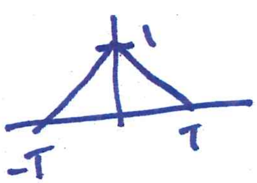
$$\delta(\tau) \leftrightarrow 1 \qquad\qquad 1 \leftrightarrow 2\pi\delta(\omega)$$

$$e^{j\beta\tau} \leftrightarrow 2\pi\delta(\omega - \beta) \qquad\qquad \cos\beta\tau \leftrightarrow \pi\delta(\omega - \beta) + \pi\delta(\omega + \beta)$$

$$e^{-\alpha|\tau|} \leftrightarrow \frac{2\alpha}{\alpha^2 + \omega^2} \qquad\qquad e^{-\alpha\tau^2} \leftrightarrow \sqrt{\frac{\pi}{\alpha}}\,e^{-\omega^2/4\alpha}$$

$$e^{-\alpha|\tau|}\cos\beta\tau \leftrightarrow \frac{\alpha}{\alpha^2 + (\omega - \beta)^2} + \frac{\alpha}{\alpha^2 + (\omega + \beta)^2}$$

$$2e^{-\alpha\tau^2}\cos\beta\tau \leftrightarrow \sqrt{\frac{\pi}{\alpha}}\left[e^{-(\omega-\beta)^2/4\alpha} + e^{-(\omega+\beta)^2/4\alpha}\right]$$



$$\begin{cases} 1 - \dfrac{|\tau|}{T} & |\tau| < T \\ 0 & |\tau| > T \end{cases} \leftrightarrow \frac{4\sin^2(\omega T/2)}{T\omega^2}$$

$$\frac{\sin\sigma\tau}{\pi\tau} \leftrightarrow \begin{cases} 1 & |\omega| < \sigma \\ 0 & |\omega| > \sigma \end{cases}$$

* This table will be included in the final exam.

Aside: A common way to estimate the instantaneous power in a signal is to cascade a memoryless system and a L.T.I. System

$$x(t) \longrightarrow \boxed{(\cdot)^2} \xrightarrow{z(t)} \boxed{\frac{1}{T}\int_{t-T}^{t} z(\alpha)\,d\alpha} \longrightarrow y(t)$$

L.T.I. System with

$$h(t) = \frac{1}{T}\,1_{[0,T]}(t)$$

Memoryless Systems

Defn: A system is called <u>memoryless</u> if its output at time t;

$$Y(t) = T[X(t)] = g(X(t)),$$

where $g: \mathbb{R} \to \mathbb{R}$, is only a function only of the current value of $X(t)$.

- $g(\cdot)$ is <u>not</u> a function of past or future values of its input
- $Y(t) = g(X(t))$ depends only on the instantaneous value of $X(t)$ at time t.

Ex. 1. $g(x) = x^2$

$$Y(t) = g(X(t)) = X^2(t)$$

is a memoryless system.

$$X(t) \longrightarrow \boxed{(\cdot)^2} \longrightarrow Y(t)$$

$X(t,\omega)$ $\qquad$ $Y(t,\omega)$

Ex. 2 Integrators are <u>not</u> memoryless. They have memory of the past.

$$Y(t) = \int_{-\infty}^{t} X(\alpha)\, d\alpha \qquad X(t) \longrightarrow \boxed{\int_{-\infty}^{t} (\cdot)\, dt} \longrightarrow Y(t)$$

$$Y(t,\omega) = \int_{-\infty}^{t} X(\alpha,\omega)\, d\alpha$$

For a memoryless system with

$$\mathbb{Y}(t) = g(\mathbb{X}(t)),$$

the first order density $f_{\mathbb{Y}(t)}(y)$ of $\mathbb{Y}(t)$ can be expressed in terms of the first order density $f_{\mathbb{X}(t)}(x)$ of $\mathbb{X}(t)$ and $g(\cdot)$

$\mathbb{X}(t)$ is just a R.V. and $\mathbb{Y}(t) = g(\mathbb{X}(t))$ is just a transformed RV (function of $\mathbb{X}(t)$.)

Note also that

$$E[\mathbb{Y}(t)] = E[g(\mathbb{X}(t))] = \int_{-\infty}^{\infty} g(x)\, f_{\mathbb{X}(t)}(x)\, dx$$

and

$$R_{YY}(t_1,t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x_1)\, g(x_2) \cdot f_{\mathbb{X}(t_1)\mathbb{X}(t_2)}(x_1,x_2)\, dx_1\, dx_2$$

Also, you can get the n-th order pdf of $\mathbb{Y}(t)$ using the mapping

$$\mathbb{Y}(t_1) = g(\mathbb{X}(t_1)),\ \mathbb{Y}(t_2) = g(\mathbb{X}(t_2)),\ \ldots,\ \mathbb{Y}(t_n) = g(\mathbb{X}(t_n)).$$

Theorem: Let $\mathbf{X}(t)$ be a S.S.S. R.P. that is the input to a memoryless system. Then the output is also a S.S.S. R.P.

Proof: Papoulis P. 394 (P. 304 in 3rd Ed.)

$$f_{\mathbf{Y}(t_1+c),\dots,\mathbf{Y}(t_n+c)}(y_1,\dots,y_n) = f_{\mathbf{X}(t_1+c),\dots,\mathbf{X}(t_n+c)}(x_1(\underline{y}),\dots,x_n(\underline{y})) \underbrace{\left|\frac{\partial(x_1,\dots,x_n)}{\partial(y_1,\dots,y_n)}\right|}_{\text{not a ftn. of } c.}$$

$$= f_{\mathbf{X}(t_1),\dots,\mathbf{X}(t_n)}(x_1(\underline{y}),\dots,x_n(\underline{y})) \left|\frac{\partial(x_1,\dots,x_n)}{\partial(y_1,\dots,y_n)}\right| = f_{\mathbf{Y}(t_1),\dots,\mathbf{Y}(t_n)}(y_1,\dots,y_n),$$

$\forall c \in \mathbb{R}$, $\forall n \in \mathbb{N}$ and all $t_1,\dots,t_n$.

n.b. Regarding the invariance of the Jacobian with a change in time origin:

Assuming $g(\cdot)$ is invertable and $h(\cdot) = g^{-1}(\cdot)$

$$x_1 = h(y_1)$$
$$x_2 = h(y_2)$$
$$\vdots$$
$$x_n = h(y_n)$$

$$\Rightarrow \frac{\partial(x_1, \ldots, x_n)}{\partial(y_1, \ldots, y_n)} = \begin{vmatrix} \frac{\partial h(y_1)}{\partial y_1} & 0 & 0 & \cdots & 0 \\ 0 & \frac{\partial h(y_2)}{\partial (y_2)} & 0 & \cdots & 0 \\ \vdots & \vdots & & & \\ 0 & 0 & 0 & \cdots & \frac{\partial h(y_n)}{\partial y_n} \end{vmatrix}$$

$$= \frac{\partial h(y_1)}{\partial y_1} \cdot \frac{\partial h(y_2)}{\partial y_2} \cdots \frac{\partial h(y_n)}{\partial y_n}.$$

Since $g(x)$ does not vary with time, neither does $h(y) = g^{-1}(y)$, so the Jacobian does not depend on time.

Example: Hard Limiter

Consider a memoryless system with

$$g(x) = \begin{cases} +1, & x \geq 0, \\ -1, & x < 0 \end{cases}$$

Consider $Y(t) = g(X(t)) = \text{sgn}(X(t))$

Find $E[Y(t)]$ and $R_{YY}(t_1, t_2)$ given the "statistics" of $X(t)$.

$$E[Y(t)] = (+1)\cdot P(\{Y(t)=1\}) + (-1)\cdot P(\{Y(t)=-1\})$$

$$= (+1)\cdot P(\{X(t)\geq 0\}) + (-1)\cdot P(\{X(t)<0\})$$

$$= 1 - F_{X(t)}(0) - F_{X(t)}(0)$$

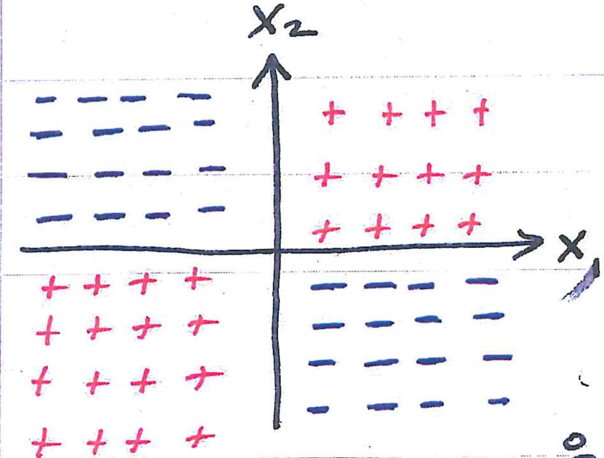
$$= 1 - 2F_{X(t)}(0).$$

$$R_{YY}(t_1,t_2) = E[\mathbb{Y}(t_1)\cdot\mathbb{Y}(t_2)]$$

<u>note</u>: $\mathbb{Y}(t_1)\cdot\mathbb{Y}(t_2) = \begin{cases} +1, & \mathbb{X}(t_1)\mathbb{X}(t_2) > 0 \\ -1, & \mathbb{X}(t_1)\mathbb{X}(t_2) \le 0 \end{cases}$

So

$$E[\mathbb{Y}(t_1)\,\mathbb{Y}(t_2)] = 1\cdot P(\{\mathbb{X}(t_1)\mathbb{X}(t_2) \ge 0\}) - 1\cdot P(\{\mathbb{X}(t_1)\mathbb{X}(t_2) < 0\})$$



where

$$P(\{\mathbb{X}(t_1)\mathbb{X}(t_2)\ge 0\}) = \int_0^\infty\int_0^\infty f_{\mathbb{X}(t_1)\mathbb{X}(t_2)}(x_1,x_2)\,dx_1\,dx_2 + \int_{-\infty}^0\int_{-\infty}^0 f_{\mathbb{X}(t_1)\mathbb{X}(t_2)}(x_1,x_2)\,dx_1\,dx_2$$

and

$$P(\{\mathbb{X}(t_1)\mathbb{X}(t_2)<0\}) = \int_{-\infty}^0\int_0^\infty f_{\mathbb{X}(t_1)\mathbb{X}(t_2)}(x_1,x_2)\,dx_1\,dx_2 + \int_0^\infty\int_{-\infty}^0 f_{\mathbb{X}(t_1)\mathbb{X}(t_2)}(x_1,x_2)\,dx_1\,dx_2$$