Session 28

ECE 600 Final Exam

Monday, April 29, 2024
3:30 to 5:30 pm
Room FNY G140 { For FNY and ONC (overflow) Sections

This will be a comprehensive exam.

This is a closed book, closed notes exam.

You may <u>not</u> use a calculator.

Start each problem on a new page.

- 10 "simple" multiple choice* questions $\boxed{50\%}$
  * There will be room to show your work.
- Two "work-out" problems $\boxed{34\%}$
- One page of True/False Questions $\boxed{16\%}$

EPE Online Section Answer Form

ECE600 Random Variables and Waveforms
Spring ~~2022~~ 2024

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MSEE 336

Final Exam Online Section Answer Form

~~May 4, 2022~~

April 29, 2024

1-10	
11	
12	
13	
Total	

$$R(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)e^{j\omega\tau}\,d\omega \leftrightarrow S(\omega) = \int_{-\infty}^{\infty} R(\tau)e^{-j\omega\tau}\,d\tau$$

$$\delta(\tau) \leftrightarrow 1 \qquad 1 \leftrightarrow 2\pi\delta(\omega)$$

$$e^{j\beta\tau} \leftrightarrow 2\pi\delta(\omega-\beta) \qquad \cos\beta\tau \leftrightarrow \pi\delta(\omega-\beta)+\pi\delta(\omega+\beta)$$

$$e^{-\alpha|\tau|} \leftrightarrow \frac{2\alpha}{\alpha^2+\omega^2} \qquad e^{-\alpha\tau^2} \leftrightarrow \sqrt{\frac{\pi}{\alpha}}\,e^{-\omega^2/4\alpha}$$

$$e^{-\alpha|\tau|}\cos\beta\tau \leftrightarrow \frac{\alpha}{\alpha^2+(\omega-\beta)^2}+\frac{\alpha}{\alpha^2+(\omega+\beta)^2}$$

$$2e^{-\alpha\tau^2}\cos\beta\tau \leftrightarrow \sqrt{\frac{\pi}{\alpha}}\,[e^{-(\omega-\beta)^2/4\alpha}+e^{-(\omega+\beta)^2/4\alpha}]$$

$$\begin{cases} 1-\dfrac{|\tau|}{T} & |\tau|<T \\ 0 & |\tau|>T \end{cases} \leftrightarrow \frac{4\sin^2(\omega T/2)}{T\omega^2}$$

$$\frac{\sin\sigma\tau}{\pi\tau} \leftrightarrow \begin{cases} 1 & |\omega|<\sigma \\ 0 & |\omega|>\sigma \end{cases}$$

Name: ______________________________

ID #: ______________________________

Directions:

1. **Print** your name and student number on the cover page.
2. Exam is closed book, closed notes, and no calculators.
3. Clearly designate all answers asked for (arrows, underline, box, etc.)

Problems 1–10 are multiple choice problems worth 5 points each. For each problem, write the lettercorresponding to the best answer next to the problem number. Space is provided to work out your solution for each of these problems. Please show your work! If your final grade is near a borderline, the quality of your written solutions could significantly impact your final grade.

1. Answer: $\boxed{C}$

$$\Phi_X(\omega) = E\left[e^{i\omega X}\right]$$

$$= \sum_{k=0}^{n} e^{i\omega k} \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \sum_{k=0}^{n} \binom{n}{k} (pe^{i\omega})^k (1-p)^{n-k}$$

Binomial Thm.

$$= (pe^{i\omega} + 1 - p)^n$$

$$= (1 + p(e^{i\omega} - 1))^n$$

etc.

2. Answer:

10. Answer:

11. Problems 11 is made up of 8 True/False questions, worth 2 points each. Fill in your answers T (true) or F (false) below, corresponding to the statements A–H in problem 11 on the exam.

A. ____

B. ____

C. ____

D. ____

E. ____

F. ____

G. ____

H. ____

Problems 12 and 13 are "work out" problems for which partial credit will be awarded for correctly reasoned work. It is important that you coherently present your thinking in the solution of these problems if you wish to receive partial credit (or full credit for that matter.) Please work problems 12 and 13 of the exam in the designated space below.

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12. Problem 12 Solution:

(Problem 12 Solution Continued)

13. Problem 13 Solution:

(Problem 13 Solution Continued)

Stochastic Processes

- The idea of a stochastic process is a straightforward extension of the idea of a random variable.

- Instead mapping each outcome $\omega \in \mathcal{S}$ to a number $X(\omega)$, we map it to a function of time $\mathbb{X}(t\ \omega)$

Stochastic Processes

Defn[(1)]**:** A random process defined on a random experiment $(\mathcal{S}, \mathcal{F}, P)$ is a family $\{X(t);\ t \in \mathbb{R}\}$ of random variables defined on $(\mathcal{S}, \mathcal{F}, P)$ and indexed by t.

Recall: A RV $X : \mathcal{S} \to \mathbb{R}$ for a particular outcome $w_0 \in \mathcal{S}$ is just a real number $X(w_0)$.

A random process $X(t, w)$ is a function of two variables: time and the outcome $w \in \mathcal{S}$.

Random process

$$\mathbf{X}(t,\omega) : \mathbb{R} \times \mathcal{S} \longrightarrow \mathbb{R}$$

real time values

outcomes of a random experiment

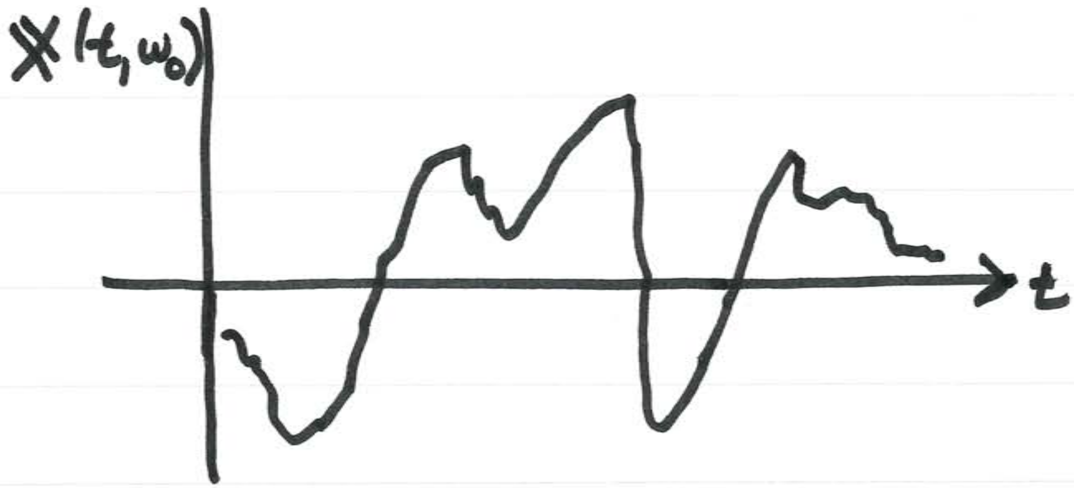
Just as we often write the random variable $\mathbf{X}(\omega)$ as $\mathbf{X}$ (hiding explicit dependence on $\omega \in \mathcal{S}$)

so we often write the random process $\mathbf{X}(t,\omega)$ as $\mathbf{X}(t)$. (again hiding the explicit dependence on $\omega \in \mathcal{S}$).

The boldface notation $\mathbf{X}$ or $\mathbf{X}(t)$ tells us there is an implicit dependence on $\omega \in \mathcal{S}$.

If we fix a particular $\omega_0 \in \mathcal{S}$ then we have

$$\mathbb{X}(\underset{\substack{\uparrow \\ t}}{\cdot}, \omega_0) : \mathbb{R} \to \mathbb{R}$$



For each $\omega \in \mathcal{S}$, $\mathbb{X}(\cdot, \omega)$ is a different function of time.

Defn: For a fixed outcome $w_0 \in \mathcal{S}$, the time function $x(\cdot) = \mathbb{X}(\cdot, w_0)$ is called the sample path of the random process $\mathbb{X}(t)$ corresponding to w_0. Each $w \in \mathcal{S}$ has a sample path $x(\cdot) = \mathbb{X}(\cdot, w)$.

The set of all sample paths

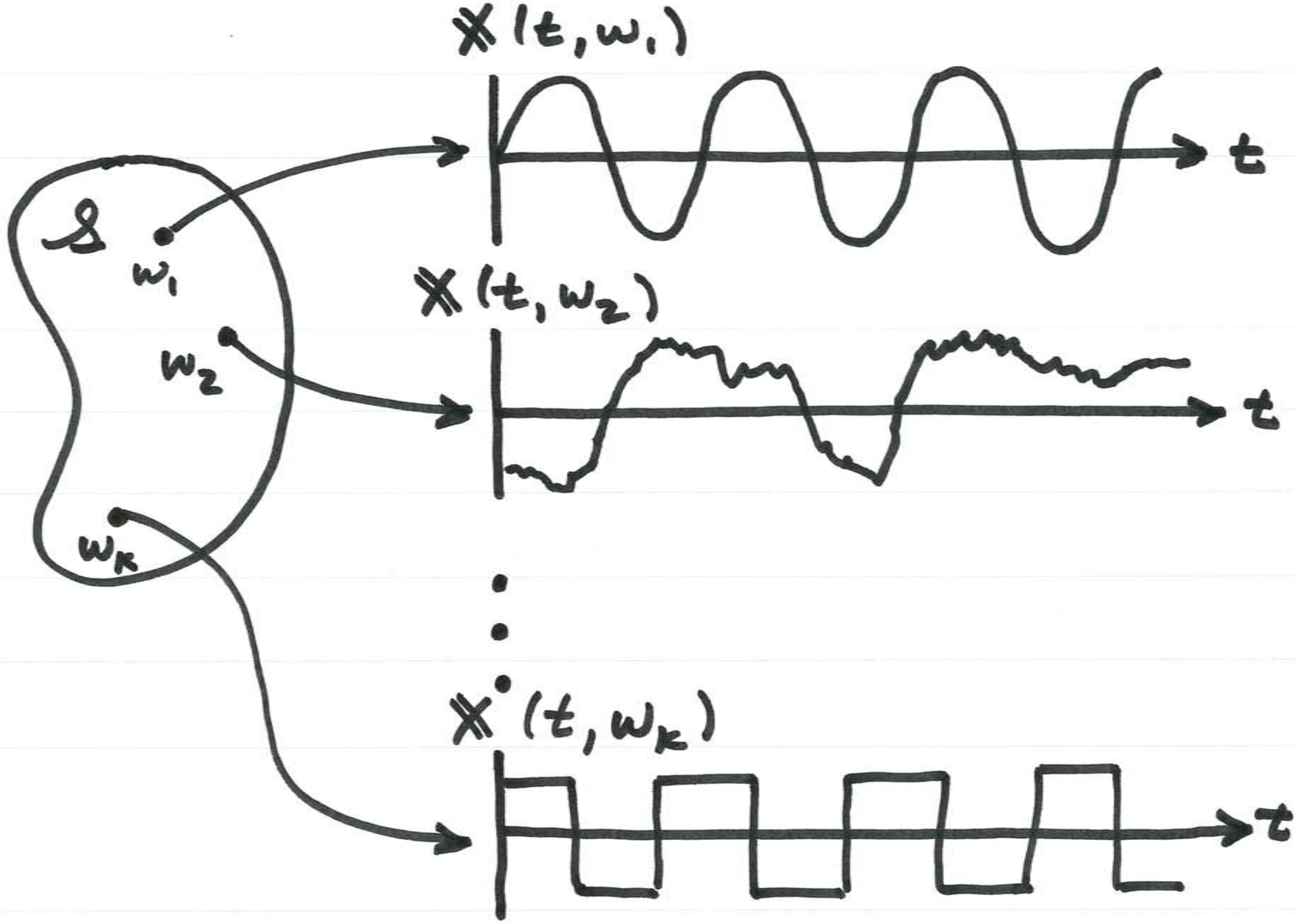
$$\mathcal{E} = \{x(\cdot) : x(\cdot) = \mathbb{X}(\cdot, w), \text{ for } w \in \mathcal{S}\}$$

is called the ensemble of the R.P. $\mathbb{X}(t)$.

Having established these ideas, we can give the following equivalent definition of a random process ...

<u>Defn</u>$^{(2)}$: Consider a random experiment $(\Omega, \mathcal{F}, P)$. A function $X : \Omega \to \mathcal{E}$, where $\mathcal{E}$ is a set of functions of time (sample paths) is called a <u>random process</u>. That is, X is a mapping assigning a function of time $X(\cdot, \omega)$ to each $\omega \in \Omega$.

A picture appears as follows:



Example: Roll a die:

$$\mathcal{S} = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6\}$$

1 2 3 4 5 6

We know how to construct $(\mathcal{S}, \mathcal{F}, P)$.

Now construct a random process $X(t)$ as

$$X(t, \omega_k) = \cos(2\pi k t), \quad k = 1, 2, \ldots, 6.$$

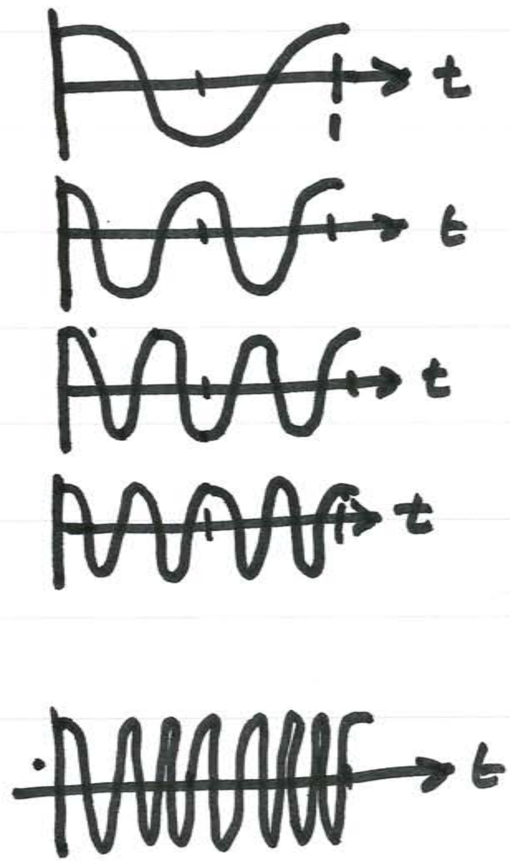
numerical outcome on die

So we now have a random process $X(t)$ that is a cosine function whose frequency is selected at random:

(example continued)

$$\mathbb{X}(t, \omega_k) = \cos(2\pi k t)$$

ω	$\mathbb{X}(t,\omega)$
ω_1	$\cos 2\pi t$
ω_2	$\cos 4\pi t$
ω_3	$\cos 6\pi t$
ω_4	$\cos 8\pi t$
$\vdots$	
ω_6	$\cos 12\pi t$



Note that once we know the particular $\omega_k \in \mathcal{S}$ that is the outcome of the experiment, there is nothing random about the random process.

Randomness only enters into the random process through the selection of $\omega \in \mathcal{S}$ in the random experiment.

<u>Note</u>: A random process is completely analogous to a Random variable:

<u>Random Variable</u> $X : \mathcal{S} \to \mathbb{R}$ (real line)

<u>Random Process</u>: $X(t) : \mathcal{S} \to \mathcal{E}$ (set of functions of time)

Example: I flip a coin three times (once a second) and construct a random process by making a waveform take on the value during that second corresponding to the outcome of the coin during that second:

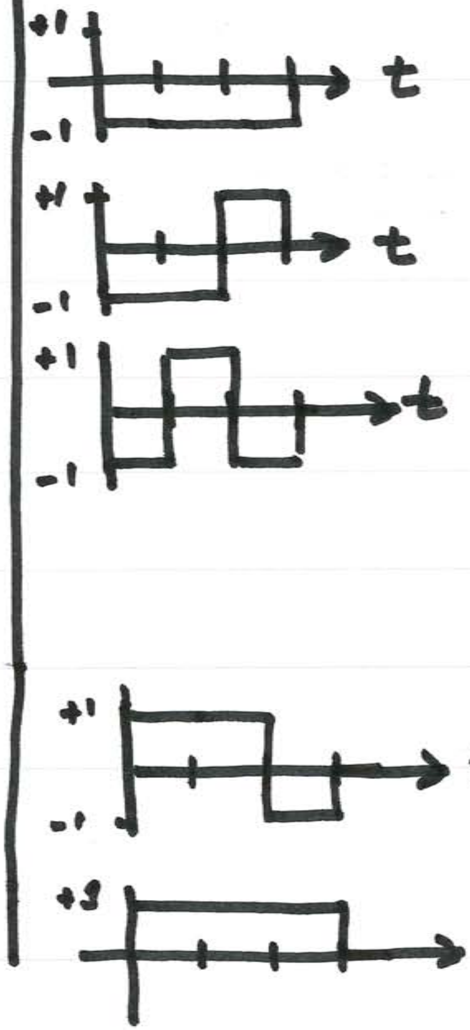
$$H \to -1 \qquad (*)$$
$$T \to +1$$

I can consider this to be a combined experiment with $2^3 = 8$ possible outcomes:

28.11

$\Omega = \{ \omega_1, \ldots, \omega_8 \}$

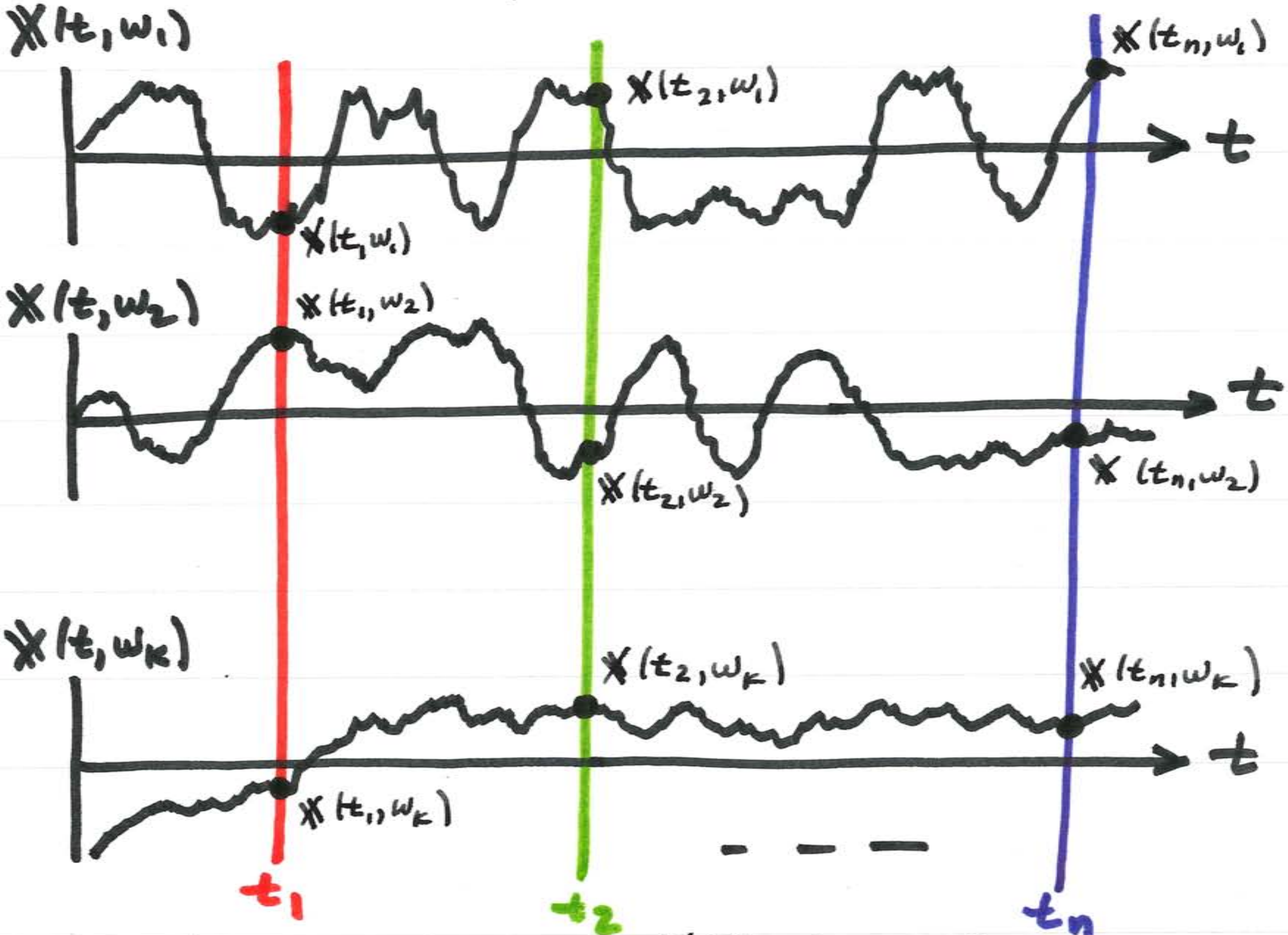
ω	
$\omega_1 = (H, H, H)$	$X(t, \omega_1)$
$\omega_2 = (H, H, T)$	$X(t, \omega_2)$
$\omega_3 = (H, T, H)$	$X(t, \omega_3)$
$\vdots$	
$\omega_7 = (T, T, H)$	$X(t, \omega_7)$
$\omega_8 = (T, T, T)$	$X(t, \omega_8)$



Randomness enters in only in the determination of ω by flipping the coin 3 times.

Once you know ω, $X(t, \omega)$ is specified by relation $(*)$ non-randomly.

Note: If we evaluate a random process at time t_1, we get a random variable $X(t_1)$:



If we sample the random process $X(t)$ at n points in time, we get n j-dist RVs $X(t_1), X(t_2), \ldots, X(t_n)$.

<u>Defn:</u> Fix n time instants $t_1, \ldots, t_n$. Then the <u>n-th order cdf</u> of $X(t)$ at times $t_1, t_2, \ldots, t_n$ is the joint cdf of the n j-dist RVs $X(t_1), X(t_2), \ldots, X(t_n)$:

$$F_{X(t_1)\cdots X(t_n)}(x_1, \ldots, x_n) = P(\{X(t_1) \le x_1\} \cap \cdots \cap \{X(t_n) \le x_n\})$$

$$= \text{" } F_{X(t)}(x_1, \ldots, x_n; t_1, \ldots, t_n) \text{ "}$$

and the corresponding <u>n-th order pdf</u> of $X(t)$ is

$$f_{X(t_1)\cdots X(t_n)}(x_1, \ldots, x_n) = \frac{\partial^n F_{X(t_1)\cdots X(t_n)}(x_1, \ldots, x_n)}{\partial x_1 \partial x_2 \cdots \partial x_n}$$

$$= \text{" } f_{X(t)}(x_1, \ldots, x_n; t_1, \ldots, t_n) \text{ "}$$

Theorem: The probabilistic behavior of a random process $X(t)$ is completely characterized by the collection of all n-th order cdfs or pdfs, for all $t_1, t_2, \ldots, t_n$, and all $n = 1, 2, 3, \ldots$ (all $n \in \mathbb{N}$.)

Proof: (Kolmogorov) Way beyond this course

But we can use the fact.

Stationary Random Processes

"Stationary" $\triangleq$ staying in one place throughout time.

What if the n-th order distributions don't change with a shift in the time origin?

" Probabilistic behavior is the same yesterday, today and tomorrow."

Probabilistic description "stays in one place" throughout time.

Defn: A random process $X(t)$ is called <u>n-th order stationary</u> if all n-th order cdfs or pdfs are invariant to time shifts in the origin:

$$f_{X(t_1)\cdots X(t_n)}(x_1,\ldots,x_n) = f_{X(t_1+c)\cdots X(t_n+c)}(x_1,\ldots,x_n)$$

for all $c \in \mathbb{R}$ and for any n time instants $t_1,\ldots,t_n$.

Defn: A random process $X(t)$ is called <u>stationary</u>, or <u>strict sense stationary</u> (SSS), if $X(t)$ is n-th order stationary for all $n = 1, 2, 3, \ldots$.

Note: If $X(t)$ is stationary, it is first-order $(n=1)$ stationary.

$$\Rightarrow \quad f_{X(t_1)}(x) = f_{X(t_1+c)}(x), \quad \forall c \in \mathbb{R}$$

$$= f_{X(t_2)}(x), \quad \text{for any } t_2 \in \mathbb{R}$$

$$= f(x) \quad (\text{not a function of time})$$

$$\therefore \quad E[X(t)] = \int_{-\infty}^{\infty} x\, f_{X(t)}(x)\, dx$$

$$= \int_{-\infty}^{\infty} x\, f(x)\, dx = \text{constant}$$

(Not a function of time.)

Similarly

$$\operatorname{var}(X(t)) = E[X^2(t)] - (E[X(t)])^2$$

$$= \sigma^2 = \text{constant}$$

(Not a function of time.)

Example: non-stationary random process.

Let $\mathbb{X}(t) \triangleq e^{-\mathbb{A}t} \cdot 1_{[0,\infty)}(t)$

where $\mathbb{A}$ is a uniformly distributed RV on $[0,1]$.

$\mathbb{X}(t)$ is a random process.

At $t = t_1$, we have that $\mathbb{X}(t_1)$ is a RV:

$$\mathbb{X}(t_1) = e^{-\mathbb{A}t_1} \cdot 1_{[0,\infty)}(t_1) = g(\mathbb{A}, t_1)$$
$$= g_{t_1}(\mathbb{A}).$$

So

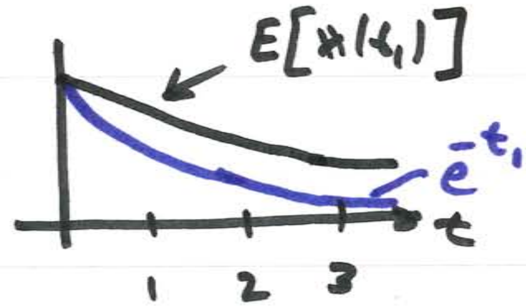
$$E[\mathbb{X}(t_1)] = E[g_{t_1}(\mathbb{A})]$$

$$= \int_{-\infty}^{\infty} g_{t_1}(a)\, f_{\mathbb{A}}(a)\, da$$

$$= \int_0^1 g_{t_1}(a) \cdot 1\, da = \int_0^1 e^{-at_1}\, da \cdot \mathbb{1}_{[0,\infty)}(t_1)$$

$$= \mathbb{1}_{[0,\infty)}(t_1)\, \frac{e^{-at_1}}{-t_1}\Bigg|_{a=0}^{1} = \frac{1-e^{-t_1}}{t_1} \cdot \mathbb{1}_{[0,\infty)}(t_1)$$

$$\therefore\ E[\mathbb{X}(t_1)] = \frac{1-e^{-t_1}}{t_1} \cdot \mathbb{1}_{[0,\infty)}(t_1)$$



Because $E[X(t)]$ is a function of t, it follows that $f_{X(t)}(x)$ must also be a function of t

(you can find $f_{X(t)}(x)$ as an exercise)

$\Rightarrow$ $X(t)$ is <u>not</u> first order stationary

$\Rightarrow$ $X(t)$ is <u>not</u> stationary

In the special case of 2nd order ($n=2$) stationary processes, we have

$$f_{X(t_1)X(t_2)}(x_1, x_2) = f_{X(t_1+c)\,X(t_2+c)}(x_1, x_2), \quad \forall c \in \mathbb{R}.$$

This means that the second-order pdf

$$f_{X(t_1)X(t_2)}(x_1, x_2) = \text{“} f_X(x_1, x_2; t_1, t_2) \text{”}$$

can depend on t_1 and t_2 only through the time difference $t_1 - t_2$.

In such cases, the correlations between $X(t_1)$ and $X(t_2)$ can depend only on the time difference $t_1 - t_2$.

This motivates the following weaker form of stationarity:

<u>Defn:</u> A random process $X(t)$ is called <u>wide-sense stationary</u> (WSS) if

(i) $E[X(t)] = \bar{X} = \text{constant}$

and

(ii) $E[X(t_1) \cdot X(t_2)] = \underbrace{R(t_1 - t_2)}$

i.e., it depends on t_1 and t_2 only through the time difference $t_1 - t_2$.

Note:

1. If a random process $X(t)$ is first and second order ($n=1,2$) stationary, and if $E[X(t)]$ and $E[X(t_1)X(t_2)]$ exist, then $X(t)$ is wide-sense stationary. The converse is <u>not</u> true.
2. If a R.P. $X(t)$ is stationary (S.S.S), it is also wide-sense stationary (W.S.S.) if $E[X(t_1)]$ and $E[X(t_1)X(t_2)]$ exist.
3. If $X(t)$ is a W.S.S R.P., $E[X(t_1)X(t_2)]$, $\mathrm{cov}(X(t_1), X(t_2))$, and the correlation coefficient $r_{X(t_1)X(t_2)}$ will depend only on the time difference $t_1 - t_2$.

Example: Consider a R.P.

$$\mathbb{X}(t) = \cos(w_0 t + \Theta)$$

where Θ is a R.V. uniformly distributed on $[0, 2\pi)$:

$$f_{\Theta}(\theta) = \begin{cases} \frac{1}{2\pi}, & 0 \leq \theta < 2\pi \\ 0, & \text{elsewhere} \end{cases}, \quad w_0 = \text{constant}$$

(radian frequency, not outcome)

Is $\mathbb{X}(t)$ W.S.S? Let's check the two defining conditions

28.26

(i) $E[X(t)] = E[\cos(\omega_0 t + \Theta)]$

$$= E[\cos(\omega_0 t)\cos(\Theta) - \sin(\omega_0 t)\sin(\Theta)]$$

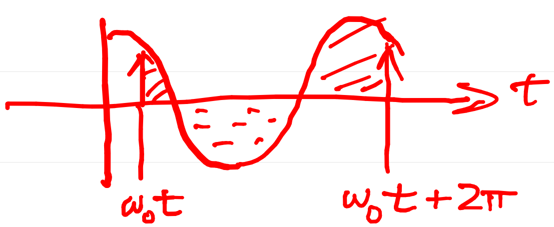
$$= \int_{-\infty}^{\infty} \left(\cos(\omega_0 t)\cos\theta - \sin(\omega_0 t)\sin(\theta)\right) f_{\Theta}(\theta)\, d\theta$$

$$= \frac{\cos \omega_0 t}{2\pi} \underbrace{\int_0^{2\pi} \cos\theta \, d\theta}_{0} - \frac{\sin \omega_0 t}{2\pi} \underbrace{\int_0^{2\pi} \sin\theta \, d\theta}_{0}$$

$= 0$ $\therefore$ $E[X(t)] = 0 =$ constant. ☑

n.b. Even easier:

$$E[X(t)] = E[\cos(\omega_0 t + \Theta)] = \frac{1}{2\pi} \underbrace{\underbrace{\int_0^{2\pi} \cos(\omega_0 t + \theta)\, d\theta}_{0}}_{\text{Integral over one period is 0.}}$$



$$\cos A \cdot \cos B = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$$

28.27

(ii) : $E[X(t_1)X(t_2)]$

$$= E\left[\cos(w_0 t_1 + \Theta)\cdot\cos(w_0 t_2 + \Theta)\right]$$

$$= E\left[\frac{1}{2}\left(\cos(w_0(t_1+t_2)+2\Theta) + \cos(w_0(t_1-t_2))\right)\right]$$

$$= \frac{1}{2}\cancelto{0}{E\left[\cos(w_0(t_1+t_2)+2\Theta)\right]} + \frac{1}{2}E\left[\cos(w_0(t_1-t_2))\right]$$

$$= \frac{1}{2}\cos(w_0(t_1-t_2))$$

$\therefore$ (i) $E[X(t)] = 0 =$ constant,

(ii) $E[X(t_1)X(t_2)] = \frac{1}{2}\cos \omega_0 (t_1 - t_2)$.

$\therefore$ $X(t)$ is a W.S.S. R.P.