Session 17

Recall...

The Characteristic Function

Defn: Let X be a R.V. on $(\mathcal{S}, \mathcal{F}, P)$.
The characteristic function of X is

$$\Phi_X(\omega) \triangleq E\left[e^{i\omega X}\right], \quad \omega \in \mathbb{R}.$$

If $f_X(x)$ is the pdf of X, then

$$\Phi_X(\omega) = \int_{-\infty}^{\infty} e^{i\omega x} f_X(x)\,dx.$$

$$\Phi_X(\omega) : \mathbb{R} \to \mathbb{C}.$$

Recall . . .

17.2

$$\Phi_X(\omega) = \int_{-\infty}^{\infty} f_X(x)\, e^{+i\omega x}\, dx$$

is a Fourier transform of the pdf $f_X(x)$

n.b., $i = \sqrt{-1}$. Engineers often use "j" instead of "i", but we will use "i".

$$\Phi_*(\omega) = \int_{-\infty}^{\infty} f_*(x)\, e^{+i\omega x}\, dx$$

n.b. $e^{i\omega x} = \cos \omega x + i \sin \omega x$ (Eulers formula)

Aside

$$e^{i\theta} = \cos\theta + i \sin\theta$$

set $\theta = \pi \Rightarrow e^{i\pi} = -1$

$$\Rightarrow e^{i\pi} + 1 = 0$$

Paul J. Nahin, Dr. Eulers Fabulous Formula, Princeton U. Press, 2006.

Also note from Euler's formula

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad \text{and} \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

From which it is easy to derive many trig formulas

$$\cos\alpha \cos\beta = \frac{1}{2}\left[\cos(\alpha-\beta) + \cos(\alpha+\beta)\right]$$

$\sin\alpha \sin\beta = \cdots,\quad \cos\alpha \cdot \sin\beta = \cdots,$

Anyway, we have

$$E[e^{i\omega X}] = E[\cos\omega X + i \sin \omega X]$$

$$= E[\cos\omega X] + i\, E[\sin \omega X]$$

n.b. $$|\Phi_X(\omega)| = \left|\int_{-\infty}^{\infty} f_X(x) e^{i\omega x} dx\right| \leq \int_{-\infty}^{\infty} |f_X(x) e^{i\omega x}| dx$$

$$= \int_{-\infty}^{\infty} \underbrace{|f_X(x)|}_{f_X(x)} \cdot \underbrace{|e^{i\omega x}|}_{1} dx \quad \int_{-\infty}^{\infty} f_X(x)\, dx = 1.$$

$$\Rightarrow |\Phi_X(\omega)| \leq \Phi_X(0) = 1.$$

$\therefore$ $\Phi_X(\omega)$ is well defined for any $f_X(x)$

There is a corresponding inverse Fourier transform relationship:

$$f_X(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_X(\omega)\, e^{-i\omega x}\, d\omega$$

$\therefore$ Given $\Phi_X(\omega)$, we can find $f_X(x)$, which is a complete probabilistic description of X

$\Rightarrow$ $\Phi_X(\omega)$ is a complete probabilistic description of X.

Fact: Suppose $\Phi_0(\omega)$ is known to be the characteristic function of RV X and RV Y:

$$\Phi_X(\omega) = \Phi_Y(\omega) \equiv \Phi_0(\omega).$$

Then X and Y have identical pdfs and cdfs:

$$f_X(\alpha) = f_Y(\alpha), \quad \forall \alpha \in \mathbb{R}$$

$$F_X(\alpha) = F_Y(\alpha), \quad \forall \alpha \in \mathbb{R}.$$

This does not mean $X(\cdot) = Y(\cdot)$.

n.b.: Just because two RVs have the same pdf does not mean that they are equal:

Let RV X have pdf

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left\{ \frac{-x^2}{2\sigma^2} \right\}$$

Define $Y = -X$. It can be shown that

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left\{ \frac{-y^2}{2\sigma^2} \right\}$$

(i.e., $f_X(\alpha) = f_Y(\alpha)$, $\forall \alpha \in \mathbb{R}$, and yet the only time $X(\omega) = Y(\omega)$ is when $X(\omega) = 0$.

$$P(\{X = Y\}) = P(\{X = 0\}) = 0$$

<u>Example:</u> If X is an exponentially distributed RV with mean μ, find $\Phi_X(\omega)$.

$$\Phi_X(\omega) = E\left[e^{i\omega X}\right] = \int_{-\infty}^{\infty} e^{i\omega x} f_X(x)\, dx$$

$$= \int_{-\infty}^{\infty} e^{i\omega x} \cdot \frac{1}{\mu} \exp\left(-\frac{x}{\mu}\right) \cdot 1_{[0,\infty)}(x)\, dx$$

$$= \int_0^{\infty} e^{i\omega x} \cdot \frac{1}{\mu} e^{-x/\mu}\, dx = \int_0^{\infty} e^{\left(i\omega - \frac{1}{\mu}\right)x}\, dx$$

$$= \cdots = \frac{\frac{1}{\mu}}{\frac{1}{\mu} - i\omega} = \frac{1}{1 - i\omega\mu} = (1 - i\omega\mu)^{-1}$$

<u>Useful Property</u>: If X is a RV with characteristic function $\Phi_X(\omega)$, and

$$Y = aX + b, \quad a, b \in \mathbb{R},$$

then $\Phi_Y(\omega) = e^{i\omega b}\,\Phi_X(a\omega)$.

<u>Proof</u>: $\Phi_Y(\omega) = E[e^{i\omega Y}] = E[e^{i\omega(aX+b)}]$

$= E[e^{i\omega a X} \cdot e^{i\omega b}] = e^{i\omega b} E[e^{i(\omega a)X}]$

$= e^{i\omega b} \cdot \Phi_X(a\omega)$

Defn: The moment generating function of a RV X is defined as

$$\phi_X(s) \triangleq E[e^{sX}],$$

where $s \in \mathbb{R}$ (or $s \in \mathbb{C}$.)

- In elementary courses, s is taken to be real
- If we take $s \in \mathbb{C}$, $\phi_X(s)$ is a bilateral Laplace transform.

$$\phi_{X}(s) \triangleq E\left[e^{sX}\right] = \int_{-\infty}^{\infty} f_X(x)\, e^{sx}\, dx$$

If we let $s \in \mathbb{C}$:

- $\Phi_X(\omega) = \phi_X(i\omega)$
- $\phi_X(\cdot) : \mathbb{C} \longrightarrow \mathbb{C}$

If $s \in \mathbb{R}$: then

$$\phi_X(\cdot) : \mathbb{R} \longrightarrow \mathbb{R}$$

Suppose I want to compute

$E[X^n]$ ~ "The n-th moment of X"

The following theorem is useful:

Moment Theorem: Given a RV X with mgf $\phi_X(s)$, the n-th moment of X is given by

$$E[X^n] = \left.\frac{d^n \phi_X(s)}{ds^n}\right|_{s=0} = \phi_X^{(n)}(0).$$

(Alternatively, we could compute $E[X^n] = \int_{-\infty}^{\infty} x^n f_X(x)\,dx$.)

Proof: Consider $\phi_X(s) = E[e^{sX}]$.

$$\phi^{(n)}(s) = \frac{d^n}{ds^n}\left(E[e^{sX}]\right) = E\left[\frac{d^n e^{sX}}{ds^n}\right]$$
$$= E[X^n e^{sX}]$$

So setting $s=0$, we have

$$\phi_X^{(n)}(0) = E[X^n \underbrace{e^{0 \cdot X}}_{1}] = E[X^n] \quad \blacksquare$$

Example: Suppose X is an exponentially distributed RV with mean μ. Find its variance.

$$\Phi_X(\omega) = \frac{1}{1-i\omega\mu} = (1-i\omega\mu)^{-1}$$

$$\phi_X(s) = \frac{1}{1-s\mu} = (1-s\mu)^{-1}$$

$$\mathrm{var}(X) = E[X^2] - (E[X])^2 = E[X^2] - \mu^2$$

$$E[X^2] \overset{m.T}{=} \left.\frac{d^2\phi_X(s)}{ds^2}\right|_{s=0}$$

$$= \frac{d^2}{ds^2}\left[(1-s\mu)^{-1}\right]\Big|_{s=0}$$

17.15

$$\ldots = \frac{d}{ds}\left[-(1-s\mu)^{-2}(-\mu)\right]\Big|_{s=0}$$

$$= \left[-2\mu\,(1-s\mu)^{-3}(-\mu)\right]\Big|_{s=0}$$

$$= 2\mu^2\,(1-s\mu)^{-3}\Big|_{s=0} = \boxed{2\mu^2}$$

$$\mathrm{var}(X) = 2\mu^2 - (\mu)^2 = \boxed{\mu^2}$$

We can use the characteristic function in the moment theorem:

$$E[X^n] = \frac{1}{i^n} \left. \frac{d^n \Phi_X(\omega)}{d\omega^n} \right|_{\omega=0}$$

<u>or</u> (identifying $\Phi_X(\omega)$ with $\phi_X(i\omega)$)

$$E[X^n] = \left. \frac{d^n \Phi_X(\omega)}{d(i\omega)^n} \right|_{i\omega=0}$$

e.g. For the exponential case, the char. ftn. is

$$\Phi_X(\omega) = (1 - i\omega\mu)^{-1}$$

$$E[X] = \frac{d}{d(i\omega)}\left[(1-i\omega\mu)^{-1}\right]\Big|_{i\omega=0}$$

$$= -(1-i\omega\mu)^{-2}(-\mu)\Big|_{i\omega=0}$$

$$= \frac{\mu}{(1-i\omega\mu)^2}\Big|_{i\omega=0} = \frac{\mu}{1} = \boxed{\mu}$$

Fact: If X is a Gaussian RV with mean μ and variance σ^2, then

$$\boxed{\Phi_X(\omega) = e^{i\omega\mu} e^{-\frac{1}{2}\omega^2\sigma^2}}$$

You should memorize this!
It will be very useful.

Recall that

$$\Phi_X(\omega) = E\left[e^{i\omega X}\right]$$

$$= \int_{-\infty}^{\infty} f_X(x)\, e^{i\omega x}\, dx$$

If X is a discrete RV taking on values $\{x_n\}$ with pmf $P_X(x_n)$, then

$$f_X(x) = \sum_n P_X(x_n)\,\delta(x - x_n),$$

and it follows that

$$\Phi_X(\omega) = \sum_n P_X(x_n)\, e^{i\omega x_n}.$$

Example: Consider a Gaussian RV X with mean μ and variance σ^2. Using the moment theorem, verify the mean and variance.

We know that $\phi_X(s) = e^{s\mu} e^{\frac{1}{2}s^2\sigma^2}$

$$E[X] = \frac{d}{ds}\phi_X(s)\Big|_{s=0} = \phi_X^{(1)}(0)$$

$$= \frac{d}{ds}\left[e^{s\mu} e^{\frac{1}{2}s^2\sigma^2}\right]$$

$$= \left(\mu e^{s\mu} \cdot e^{\frac{1}{2}s^2\sigma^2} + e^{s\mu}\sigma^2 s\, e^{\frac{1}{2}s^2\sigma^2}\right)\Big|_{s=0}$$

$$= \mu \cdot 1 \cdot 1 + 1 \cdot 0 \cdot 1 = \boxed{\mu}$$

17.21

Now $\text{var}(X) = E[X^2] - (E[X])^2$

$$E[X^2] = \phi_X^{(2)}(0)$$

$$= \frac{d}{ds}\left(\mu e^{\mu s} \cdot e^{\frac{1}{2}s^2\sigma^2} + e^{\mu s}\sigma^2 s e^{\frac{1}{2}\sigma^2 s^2}\right)$$

$$= \Big(\mu^2 e^{\mu s} \cdot e^{\frac{1}{2}s^2\sigma^2} + \mu e^{\mu s} s\sigma^2 e^{1/2 s^2\sigma^2}$$

$$+ e^{\mu s}\sigma^2 e^{\frac{1}{2}\sigma^2 s^2} + s\sigma^2(\mu + \sigma^2 s)e^{\mu s}e^{\frac{1}{2}\sigma^2 s^2}\Big)\Big|_{s=0}$$

$$= \mu^2 \cdot 1 \cdot 1 + \mu \cdot 1 \cdot 0 \cdot \sigma^2 \cdot 1 + 1 \cdot \sigma^2 \, 1$$

$$+ 0 \cdot (\mu + 0) \cdot 1 \cdot 1 = \boxed{\mu^2 + \sigma^2}$$

$$\therefore \text{Var}(X) = \mu^2 + \sigma^2 - (\mu)^2 = \sigma^2.$$

Exam 2 Cut-off