Session 16

Recall...

Mean, Variance and Expectation

16.1

Defn: The mean or expected value of a RV X with pdf $f_X(x)$ is

$$E[X] \triangleq \int_{-\infty}^{\infty} x f_X(x)\, dx.$$

Recall...

$\therefore$ For a discrete RV X, we have

$$\boxed{E[X] = \sum_k x_k \, p_X(x_k).}$$

If you know about Riemann-Stiltjes integrals, you can write

Don't worry about this (Riemann-Stiltjes)

$$E[X] = \int_{-\infty}^{\infty} x \, dF_X(x)$$

$$= \begin{cases} \sum_k x_k \, p_X(x_k) & (\text{discrete RV } X) \\ \int_{-\infty}^{\infty} x \, f_X(x) \, dx & (\text{continuous RV } X) \end{cases}$$

Recall...

16.3

<u>Defn</u>: Let $\mathbb{X}$ be a RV on $(\mathcal{S}, \mathcal{F}, P)$ and let $M \in \mathcal{F}$. Then the <u>conditional mean of $\mathbb{X}$ conditioned on M</u> is

$$E[\mathbb{X}|M] \triangleq \int_{-\infty}^{\infty} x \, f_{\mathbb{X}}(x|M) \, dx .$$

(<u>n.b.</u> If $\mathbb{X}$ is discrete, we have the conditional pmf $p_{\mathbb{X}}(x_k|M) = P(\{\mathbb{X}=x_k\}|M)$, and then

$$E[\mathbb{X}|M] = \int_{-\infty}^{\infty} x f_{\mathbb{X}}(x|M)\, dx = \int_{-\infty}^{\infty} x \left(\sum_k p_{\mathbb{X}}(x_k|M)\, \delta(x-x_k) \right) dx$$

$$= \sum_k p_{\mathbb{X}}(x_k|M) \cdot \int_{-\infty}^{\infty} x \, \delta(x-x_k)\, dx = \sum_k x_k \, p_{\mathbb{X}}(x_k|M) \Big)$$

Recall...

Example: Let X be an exponentially distributed RV with pdf

$$f_X(x) = \frac{1}{\mu} \exp\left\{-\frac{x}{\mu}\right\} \mathbb{1}_{[0,\infty)}(x), \quad \mu > 0$$

What is $E[X]$?

$$E[X] = \int_{-\infty}^{\infty} x\, f_X(x)\, dx = \int_{0}^{\infty} x \cdot \frac{1}{\mu} e^{-x/\mu}\, dx$$

$$\underset{\text{(exercise)}}{\overset{\text{int. by parts}}{=}} \cdots = \left[-x e^{-x/\mu} - \mu e^{-x/\mu}\right]_0^{\infty} = \boxed{\mu}$$

Now let's consider the conditional mean

$$E[X \mid M], \quad \text{where } M = \{X > \mu\}.$$

$$E[\mathbb{X} \mid \{\mathbb{X} > \mu\}] = \int_{-\infty}^{\infty} x \, f_{\mathbb{X}}(x \mid \{\mathbb{X} > \mu\}) \, dx$$

It is straightforward to show that

$$f_{\mathbb{X}}(x \mid \{\mathbb{X} > \mu\}) = \frac{\frac{1}{\mu} \exp\left(\frac{-x}{\mu}\right) \cdot 1_{(\mu,\infty)}(x)}{e^{-\mu/\mu}}$$

(Exercise)

$$= \frac{1}{\mu} \exp\left\{-\frac{(x-\mu)}{\mu}\right\} \cdot 1_{(\mu,\infty)}(x)$$

$$\therefore \; E[\mathbb{X} \mid \{\mathbb{X} > \mu\}] = \int_{-\infty}^{\infty} x \, f_{\mathbb{X}}(x \mid \{\mathbb{X} > \mu\}) \, dx$$

$$= \int_{\mu}^{\infty} x \cdot \frac{1}{\mu} \exp\left\{\frac{-(x-\mu)}{\mu}\right\} dx$$

let $r = x - \mu$, $x = r + \mu$, $dx = dr$ $\Rightarrow$ $\displaystyle = \int_{0}^{\infty} (r+\mu) \cdot \frac{1}{\mu} \exp\left\{\frac{-r}{\mu}\right\} dr = \cdots$

$$= \underbrace{\int_0^\infty \frac{r}{\mu} \exp\left(\frac{-r}{\mu}\right) dr}_{\mu} + \mu \underbrace{\int_0^\infty \frac{1}{\mu} \exp\left(\frac{-r}{\mu}\right) dr}_{\mu \cdot 1}$$

$$= \boxed{2\mu}$$

n.b. $E[X] \neq E[X \mid \{X > \mu\}]$

More generally

$$E[X \mid M] \neq E[X].$$

Suppose we have a RV X defined on $(\mathcal{S}, \mathcal{F}, P)$ and having pdf $f_X(x)$

Now suppose I have a new RV

$$Y = g(X)$$

where $g : \mathbb{R} \to \mathbb{R}$. What is $E[Y]$?

It appears we must first find $f_Y(y)$, and then compute

$$E[Y] = \int_{-\infty}^{\infty} y \, f_Y(y) \, dy .$$

This will work, but there is an easier way.

<u>Fact</u>: Let $\mathbb{X}$ be a RV and define the new RV $\mathbb{Y} = g(\mathbb{X})$.

Then

$$E[\mathbb{Y}] = E[g(\mathbb{X})] = \int_{-\infty}^{\infty} g(x) f_{\mathbb{X}}(x)\, dx.$$

<u>n.b.</u> $\displaystyle\int_{-\infty}^{\infty} g(x) f_{\mathbb{X}}(x)\, dx = \int_{-\infty}^{\infty} y\, f_{\mathbb{Y}}(y)\, dy$

<u>Proof</u>: Outlined in Papoulis.

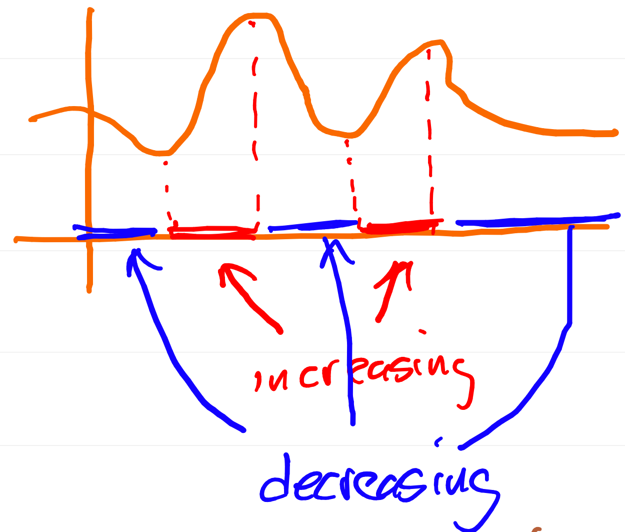
Basic idea: Assume $g(x)$ is monotonically increasing

$$y = g(x), \quad \frac{dx}{dy} = \left|\frac{dx}{dy}\right| > 0$$

So
$$E[Y] \triangleq \int_{-\infty}^{\infty} y\, f_Y(y)\, dy$$

$$= \int_{-\infty}^{\infty} \underbrace{g(x)}_{y} \cdot \underbrace{f_X(x) \frac{dx}{\cancel{dy}}}_{f_Y(y)} \cancel{dy}$$

$$= \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$$



A general function can be broken into monotonically increasing and decreasing segments (and flat segments)

Similar for monotonically decreasing function

(You are not responsible for proof.)

"**Defn:**" The expected value of a function $g(X)$ of a RV X is

$$E[g(X)] \triangleq \int_{-\infty}^{\infty} g(x) f_X(x)\,dx$$

n.b. If X is a discrete RV, this becomes

$$E[g(X)] = \sum_k g(x_k)\, p_X(x_k)$$

Linearity of Expectation

Let $g_1(X)$ and $g_2(X)$ be two functions of a RV X and let α and β be two constants ($\alpha, \beta \in \mathbb{R}$ or $\mathbb{C}$).

Then

$$E[\alpha g_1(X) + \beta g_2(X)]$$

$$= \alpha E[g_1(X)] + \beta E[g_2(X)]$$

Proof: Exercise

Defn: The variance of a RV X is defined as

$$\operatorname{var}(X) \triangleq E[(X-\overline{X})^2]$$

$$= \int_{-\infty}^{\infty} (x-\overline{X})^2 f_X(x)\, dx ,$$

where $\overline{X} = E[X]$

Defn: The positive square root of the variance of X is called the standard deviation of X:

$$\operatorname{StDev}(X) = \sigma_X = \sqrt{\operatorname{var}(X)}$$

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n.b. $\mathrm{var}(X) \triangleq E[(X-\bar{X})^2]$

$$= E[(X^2 - 2\bar{X}X + (\bar{X})^2]$$

$$= E[X^2] - 2\bar{X}E[X] + (\bar{X})^2$$

$$= E[X^2] - 2\bar{X}\cdot\bar{X} + (\bar{X})^2$$

$$= E[X^2] - 2(\bar{X})^2 + (\bar{X})^2$$

$$= E[X^2] - (E[X])^2$$

$$\therefore \boxed{\mathrm{Var}(X) = E[X^2] - (E[X])^2}$$

Ex. 1: Consider a Gaussian RV X with pdf

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left\{ -\frac{(x-\mu)^2}{2\sigma^2} \right\},$$

where $\mu \in \mathbb{R}$ and $\sigma > 0$.

Mean:

$$E[X] = \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi}\,\sigma} \exp\left\{ -\frac{(x-\mu)^2}{2\sigma^2} \right\} dx$$

Let $r = x - \mu \Rightarrow x = r + \mu \Rightarrow dx = dr$

$$= \int_{-\infty}^{\infty} \frac{r+\mu}{\sqrt{2\pi}\,\sigma} \exp\left\{ \frac{-r^2}{2\sigma^2} \right\} dr$$

$$= \int_{-\infty}^{\infty} \frac{r}{\sqrt{2\pi}\,\sigma} \exp\left\{ \frac{-r^2}{2\sigma^2} \right\} dr + \mu \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left\{ \frac{-r^2}{2\sigma^2} \right\} dr$$

$$= 0 + \mu \cdot 1 = \boxed{\mu}$$

variance: $\mathrm{var}(X) = E[X^2] - \mu^2$

$$\underline{E[X^2]} = \int_{-\infty}^{\infty} \underline{\underline{x^2}} \cdot \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{ -\frac{(x-\mu)^2}{2\sigma^2} \right\} dx$$

let $r = x - \mu$

$x = r + \mu$

$dx = dr$

$$= \int_{-\infty}^{\infty} (r+\mu)^2 \cdot \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{ \frac{-r^2}{2\sigma^2} \right\} dr$$

(exercise)

$$= \cdots = \sigma^2 + \mu^2$$

$$\Rightarrow \quad \mathrm{var}(X) = \sigma^2 + \mu^2 - (\mu)^2 = \boxed{\sigma^2}$$

$\therefore$ A Gaussian RV with pdf

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{ -\frac{(x-\mu)^2}{2\sigma^2} \right\}$$

has mean μ and variance σ^2

Ex. 2 Consider the Poisson RV X with pmf

$$P_X(k) = p_k = P(\{X = k\})$$

$$= \frac{e^{-\mu}\mu^k}{k!}, \quad k = 0, 1, 2, \ldots, \quad \mu > 0$$

Compute the mean an variance of X.

mean: $$E[X] = \sum_{\cancel{k=0}\ k=1}^{\infty} \cancel{k} \frac{e^{-\mu}\mu^k}{\cancel{k!}\ (k-1)!} = \sum_{k=1}^{\infty} \frac{e^{-\mu}\mu^k}{(k-1)!}$$

$$= \mu e^{-\mu} \sum_{k=1}^{\infty} \frac{\mu^{k-1}}{(k-1)!} = e^{-\mu}\mu \underbrace{\sum_{m=0}^{\infty} \frac{\mu^m}{m!}}_{e^{\mu}}$$

$$= e^{-\mu} \cdot \mu \cdot e^{\mu} = \boxed{\mu}$$

variance: $\operatorname{var}(X) = E[X^2] - (E[X])^2$

$$= E[X^2] - \mu^2$$

So $$E[X^2] = \sum_{\cancel{k=0}\; k=1}^{\infty} \frac{k^2 e^{-\mu} \mu^k}{k!} = \sum_{k=1}^{\infty} \frac{k e^{-\mu} \mu^t}{(k-1)!}$$

$s = k-1$
$\Rightarrow k = s+1$

$$= \sum_{s=0}^{\infty} \frac{(s+1) e^{-\mu} \mu^{s+1}}{s!}$$

$$= \mu \underbrace{\sum_{s=0}^{\infty} \frac{s e^{-\mu} \mu^s}{s!}}_{\mu} + \mu \underbrace{\sum_{k=0}^{\infty} \frac{e^{-\mu} \mu^s}{s!}}$$

$$= \mu \cdot \mu + \mu \cdot 1 = \mu^2 + \mu$$

$$\therefore \operatorname{var}(X) = \mu^2 + \mu - \mu^2 = \boxed{\mu}$$

The Characteristic Function

Defn: Let $\mathbb{X}$ be a R.V. on $(\mathcal{S}, \mathcal{F}, P)$. The characteristic function of $\mathbb{X}$ is

$$\Phi_{\mathbb{X}}(\omega) \triangleq E\left[e^{i\omega \mathbb{X}}\right], \quad \omega \in \mathbb{R}.$$

If $f_{\mathbb{X}}(x)$ is the pdf of $\mathbb{X}$, then

$$\Phi_{\mathbb{X}}(\omega) = \int_{-\infty}^{\infty} e^{i\omega x} f_{\mathbb{X}}(x)\, dx.$$

$$\Phi_{\mathbb{X}}(\omega) : \mathbb{R} \rightarrow \mathbb{C}.$$

$$\Phi_X(\omega) = \int_{-\infty}^{\infty} f_X(x)\, e^{+i\omega x}\, dx$$

is a Fourier transform of the pdf $f_X(x)$

<u>n.b.</u>, $i = \sqrt{-1}$. Engineers often use "j" instead of "i", but we will use "i".