Session 14

Recall...

The Total Prob. Law and Bayes Theorem

14.1

Given a RV X on $(S, \mathcal{F}, P)$, let $\{A_1, \ldots, A_n\}$ be a partition of S, with $A_k \in \mathcal{F}$, $k = 1, \ldots, n$.

Then

$$P(\{X \le x\}) = P(\{X \le x\} | A_1) P(A_1) + P(\{X \le x\} | A_2) P(A_2) + \cdots + P(\{X \le x\} | A_n) P(A_n) \quad \text{--- (1)}$$

But note that

$$P(\{X \le x\}) = F_X(x)$$

$$P(\{X \le x\} | A_k) = F_X(x | A_k)$$

Recall...

14.2

$$\therefore \quad F_X(x) = F_X(x|A_1)P(A_1) + F_X(x|A_2)P(A_2) + \cdots + F_X(x|A_n)P(A_n) \qquad \text{--- (1A)}$$

note $\quad f_X(x) = \dfrac{dF_X(x)}{dx}$

and $\quad f_X(x|A_k) = \dfrac{dF_X(x|A_k)}{dx}$

$$\Rightarrow \quad f_X(x) = f_X(x|A_1)P(A_1) + f_X(x|A_2)P(A_2) + \cdots + f_X(x|A_n)P(A_n) \qquad \text{--- (1B)}$$

Recall...

2. Recall Bayes Formula:

$$P(A|B) = \frac{P(B|A)\,P(A)}{P(B)}.$$

Let $B = \{X \le x\}$. Then

$$P(A|\{X \le x\}) = \frac{P(\{X \le x\}|A)\,P(A)}{P(\{X \le x\})}$$

$$= \frac{F_X(x|A)\,P(A)}{F_X(x)}$$

$$\therefore \boxed{P(A|\{X \le x\}) = \frac{F_X(x|A)\,P(A)}{F_X(x)}} \quad \text{--- (2)}$$

Now consider $P(A|B) = \dfrac{P(B|A)\,P(A)}{P(B)}$

Let $B = \{x_1 < X \le x_2\}$, $x_1 < x_2$

Then we have

$$P(A \mid \{x_1 < X \le x_2\}) = \frac{P(\{x_1 < X \le x_2\} \mid A)\,P(A)}{P(\{x_1 < X \le x_2\})}$$

$$= \frac{\left(F_X(x_2|A) - F_X(x_1|A)\right) P(A)}{F_X(x_2) - F_X(x_1)} \quad \text{---} \quad (3)$$

What about $P(A \mid \{X = x\})$

Suppose $B = \{X = x\}$. What is $P(A \mid B)$?

$$P(A \mid \{X = x\}) \stackrel{?}{=} \frac{P(A \cap \{X = x\})}{P(\{X = x\})}$$

n.b. $P(\{X = x\}) = 0$ if RV X is continuous

and $A \cap \{X = x\} \subset \{X = x\}$

$$\Rightarrow \underbrace{P(A \cap \{X = x\})}_{=\,0} \leq P(\{X = x\}) = 0$$

If X is abs. continuous

So we have

$$P(A \mid \{X = x\}) = \frac{P(A \cap \{X = x\})}{P(\{X = x\})} = \frac{0}{0} : \text{undefined.}$$

But clearly the probability $P(A \mid \{X = x\})$ is meaningful.

Approach:

Consider $P(A \mid \{x_1 < X \leq x_2\})$

where $x_1 = x$

$x_2 = x + \Delta x$

Then we consider

$$\lim_{\Delta x \to 0} P(A \mid \{x < X \leq x + \Delta x\}) = P(A \mid \{X = x\})$$

So

$$P(A \mid \{X = x\}) = \lim_{\Delta x \to 0} P(A \mid \{x < X \le x + \Delta x\})$$

$$\overset{(3)}{=} \lim_{\Delta x \to 0} \left[\frac{F_X(x+\Delta x \mid A) - F_X(x \mid A)}{F_X(x+\Delta x) - F_X(x)} \right] P(A)$$

$$= \lim_{\Delta x \to 0} \left[\frac{\dfrac{F_X(x+\Delta x \mid A) - F_X(x \mid A)}{\Delta x}}{\dfrac{F_X(x+\Delta x) - F_X(x)}{\Delta x}} \right] P(A)$$

Aside:

Fact: $\lim_{p\to 0} \frac{A(p)}{B(p)} = \frac{\left(\lim_{p\to 0} A(p)\right)}{\left(\lim_{q\to 0} B(q)\right)}$

Assuming $\lim_{p\to 0} A(p)$, $\lim_{p\to 0} B(p)$ and $\lim_{p\to 0} \frac{A(p)}{B(p)}$ are well defined

$$= \cdots = \frac{f_X(x|A)}{f_X(x)} P(A).$$

$$\therefore \quad \boxed{P(A \mid \{X = x\}) = \frac{f_X(x \mid A)}{f_X(x)} P(A)} \quad (4)$$

From (4), multiplying both sides by $f_X(x)$, we get

$$P(A \mid \{X = x\}) f_X(x) = f_X(x \mid A) P(A)$$

$$\Rightarrow \int_{-\infty}^{\infty} P(A \mid \{X = x\}) f_X(x)\, dx = P(A) \underbrace{\int_{-\infty}^{\infty} f_X(x \mid A)\, dx}_{1}$$

$$\Rightarrow P(A) = \int_{-\infty}^{\infty} P(A \mid \{X = x\}) f_X(x)\, dx.$$

$$\therefore \quad \boxed{P(A) = \int_{-\infty}^{\infty} P(A \mid \{\cancel{X} = x\}) f_X(x)\, dx}$$

Total Probability Law --- [5]

Furthermore from [4]

$$f_{X}(x|A) = \frac{P(A|\{X=x\})\, f_X(x)}{P(A)}$$

$$\therefore \boxed{f_{X}(x|A) = \frac{P(A|\{X=x\})\, f_X(x)}{\int_{-\infty}^{\infty} P(A|\{X=\alpha\})\, f_X(\alpha)\, d\alpha}}$$

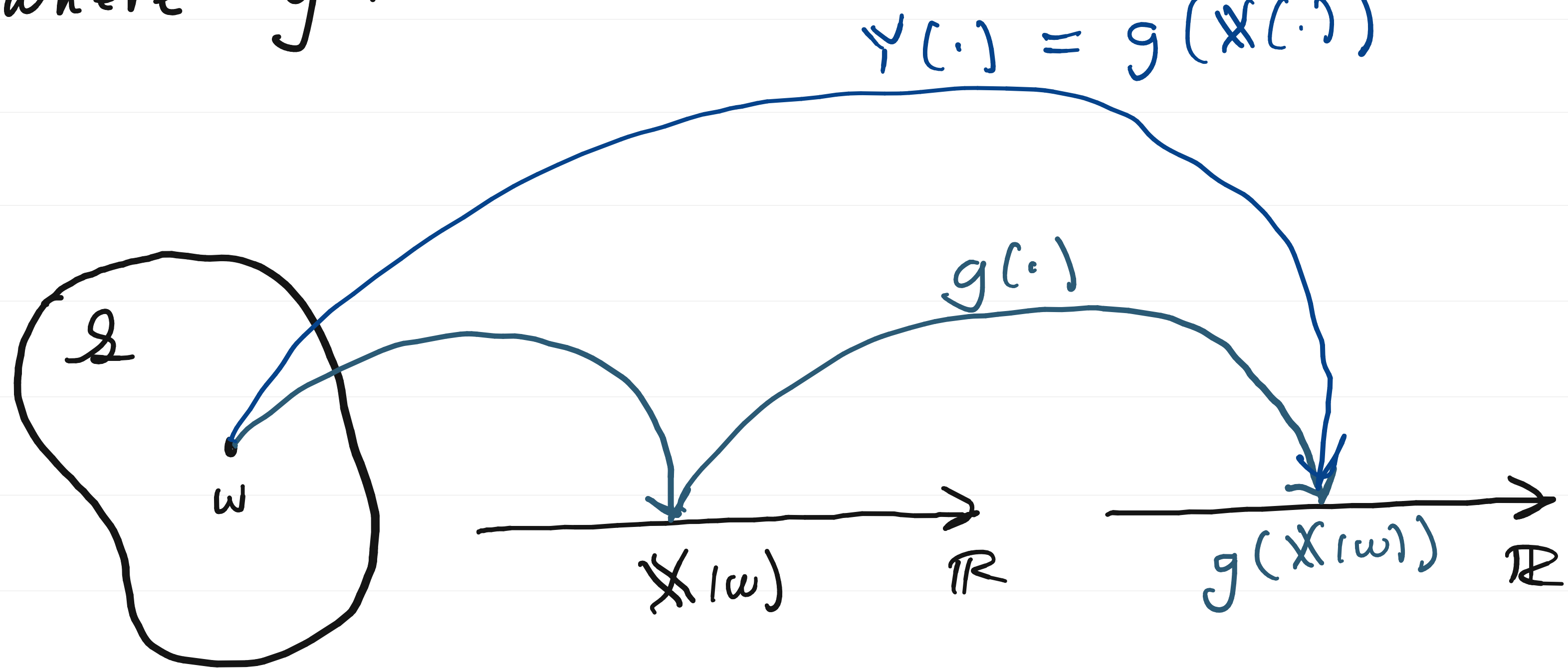
Bayes Theorem --- [6]

Functions of a Random Variable

Assume X is a RV on $(S, \mathcal{F}, P)$. Now assume we have a function

$$Y = g(X),$$

where $g : \mathbb{R} \to \mathbb{R}$



So $\mathbb{Y}(\cdot) = g(\mathbb{X}(\cdot)) : \mathcal{S} \to \mathbb{R}$.

It appears to be a random variable.

Is it ?

Recall: From the defn. of a RV, $\mathbb{Y} : \mathcal{S} \to \mathbb{R}$ is a RV if

$$\mathbb{Y}^{-1}(A) = \{\omega \in \mathcal{S} : Y(\omega) \in A\} \in \mathcal{F},\ \forall A \in \mathbb{B}(\mathbb{R}),$$

where $\mathcal{F}$ is the event space of $(\mathcal{S}, \mathcal{F}, P)$.

For $\mathbb{Y} = g(\mathbb{X})$ to be measurable (i.e, an RV) $g(\cdot)$ must satisfy the following properties:

1. The domain of $g(\cdot)$ must contain the range space of $\mathbb{X}$.

2. For each $y \in \mathbb{R}$, the set $R_y \triangleq \{x \in \mathbb{R} : g(x) \leq y\}$ must be a Borel set.

3. The events $\{g(\mathbb{X}) = \pm\infty\}$ must have probability 0.

Any function $g(\cdot)$ satisfying these 3 properties is called a Baire function

For such functions $g(\cdot)$,

$$\mathbb{Y} = g(\mathbb{X})$$

is a valid random variable.

All functions we typically encounter in engineering are Baire functions.

The Distribution of $\mathbb{Y} = g(\mathbb{X})$

Given a RV $\mathbb{X}$ on $(\mathcal{S}, \mathcal{F}, P)$ with cdf $F_{\mathbb{X}}(x)$, and given a function $g: \mathbb{R} \to \mathbb{R}$, we define $\mathbb{Y} = g(\mathbb{X})$.

Find

$$F_{\mathbb{Y}}(y) = P(\{\mathbb{Y} \leq y\}) = P(\{g(\mathbb{X}) \leq y\})$$

$$= P(\{\mathbb{X} \in g^{-1}((-\infty, y]))$$

$$= P_X(g^{-1}(-\infty, y])$$

This tells us how to find $F_{\mathbb{Y}}(y)$ for any $y \in \mathbb{R}$.

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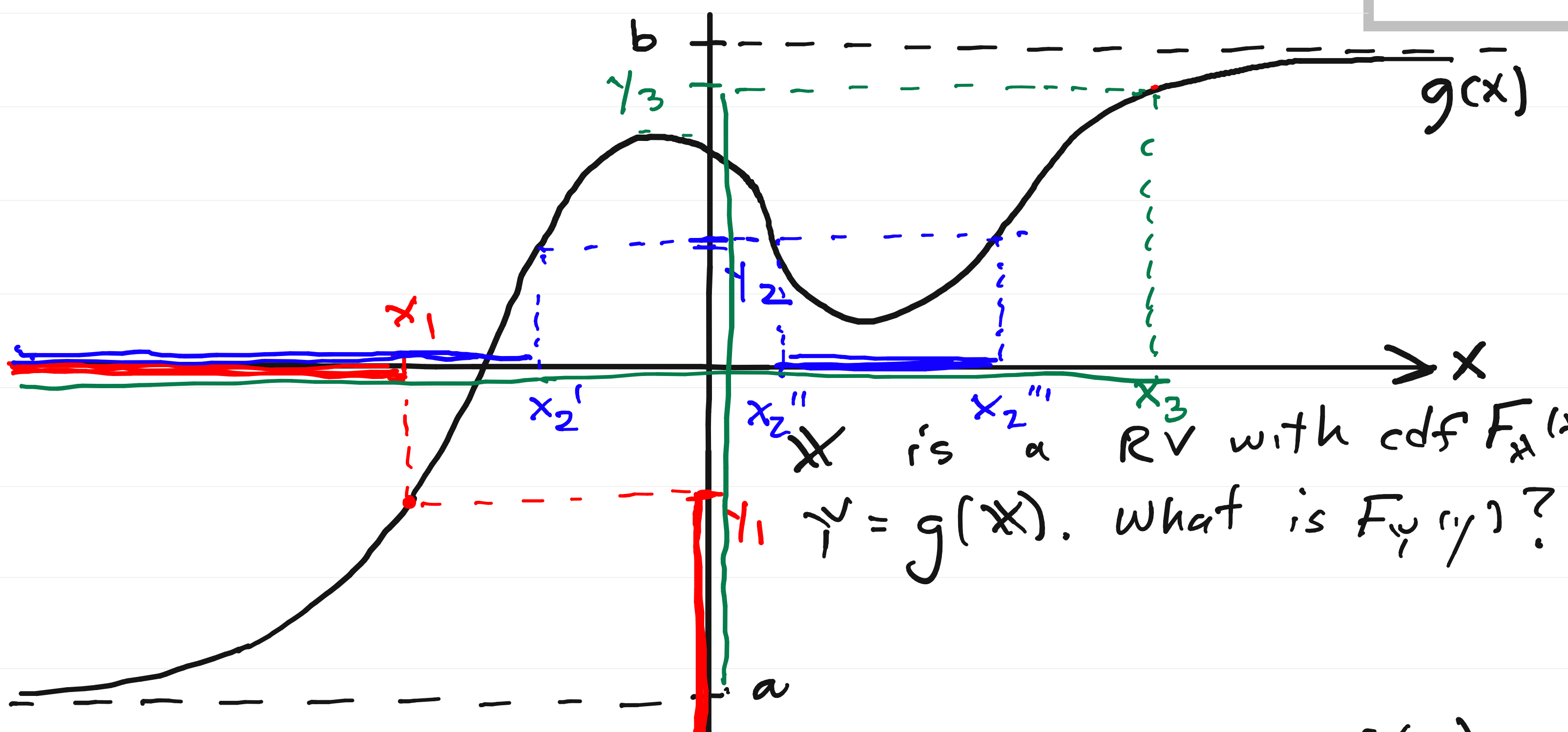
1. Consider a generic function $g(\cdot)$:



$\mathbb{X}$ is a RV with cdf $F_{\mathbb{X}}(x)$. $\mathbb{Y} = g(\mathbb{X})$. What is $F_{\mathbb{Y}}(y)$?

Consider y_1: $F_{\mathbb{Y}}(y_1) = P[\{\mathbb{Y} \le y_1\}] = P[\{\mathbb{X} \le x_1\}] = F_{\mathbb{X}}(x_1)$

Consider y_2: $F_{\mathbb{Y}}(y_2) = P[\{\mathbb{Y} \le y_2\}] = P(\{\mathbb{X} \le x_2'\} \cup \{x_2'' < \mathbb{X} \le x_2'''\})$

disjoint

$$\therefore\ F_Y(y_2) = P(\{X \le x_2'\}) + P(\{x_2'' \le X \le x_2'''\})$$
$$= F_X(x_2') + (F_X(x_2''') - F_X(x_2''))$$

If X is ~~not~~ absolutely continuous, we have to think carefully about the situation at $x = x_2''$.

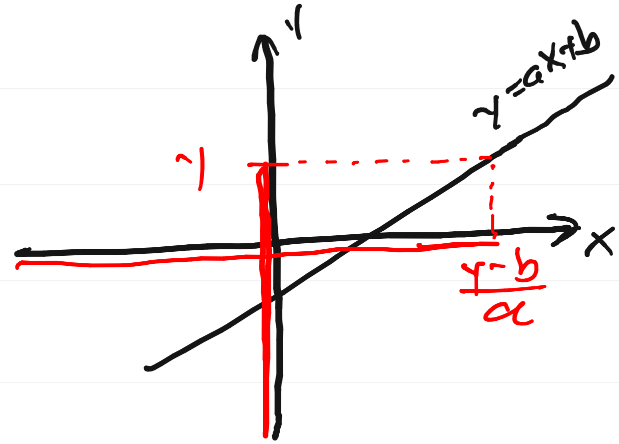
So to determine $F_Y(y)$ completely, we must do this for all $y \in \mathbb{R}$.

This can be easy or difficult depending the complexity of $g(\cdot)$

Ex. A $\quad Y = aX + b, \quad a, b \in \mathbb{R}$

$\Rightarrow \quad g(x) = ax + b$ (affine transformation)

two cases: $\quad a \gtrless 0$.



(i) $a > 0$

$$F_Y(y) = P(\{Y \le y\}) = P(\{aX + b\} \le y\})$$
$$= P(\{X \le \frac{y-b}{a}\})$$
$$= F_X\left(\frac{y-b}{a}\right)$$

$$f_Y(y) = \frac{dF_Y(y)}{dy} = f_X\left(\frac{y-b}{a}\right) \cdot \frac{1}{a}$$
$$= \frac{1}{a} f_X\left(\frac{y-b}{a}\right)$$

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(ii) $\underline{a < 0}$: $F_Y(y) = P(\{Y \le y\})$

$= P(\{aX + b \le y\})$

$= P(\{X \ge \frac{y-b}{a}\})$

$= P(\{X > \frac{y-b}{a}\})$

$= 1 - F_X\left(\frac{y-b}{a}\right)$