Session 13

Recall...

13.1

Ex. 1 Gaussian RV

A RV X is Gaussian if it has a pdf of the form

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}, \quad \forall x \in \mathbb{R}$$

where $\mu \in \mathbb{R}$ and $\sigma > 0$.

n.b. $$F_X(x) = \int_{-\infty}^{x} f_X(\alpha)\, d\alpha = \Phi\left(\frac{x-\mu}{\sigma}\right) \quad \left(G\left(\frac{x-\mu}{\sigma}\right) \text{ in Papoulis}\right)$$

$\mathcal{N}[\mu, \sigma^2]$

where $$\Phi(r) = \int_{-\infty}^{r} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz.$$

Recall...

13.2

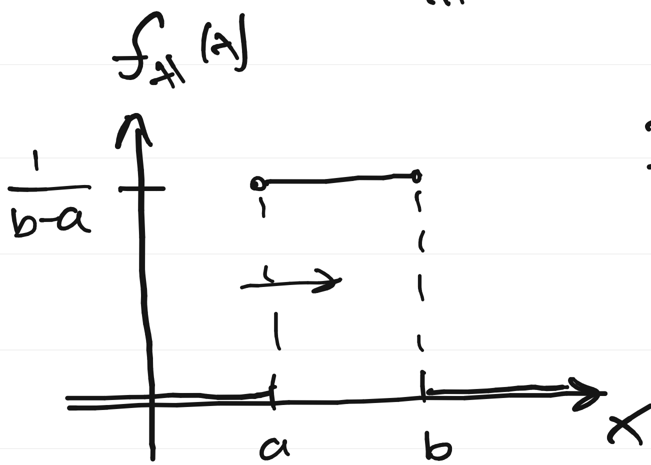
Ex. 2 Uniformly Distributed RV

A RV has a uniform distribution,

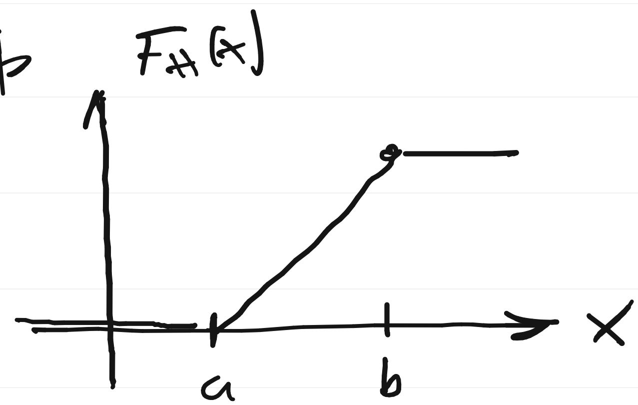
$$X \sim U[a,b], \quad a<b$$

if

$$f_X(x) = \frac{1}{b-a} \cdot 1_{[a,b]}(x)$$



Integrate ⇒



Ex. 3 Binomially Distributed RV

A binomially distributed RV is a discrete RV taking on values in the set $\{0, 1, 2, \ldots, n\} \subset \mathbb{R}$ with pmf

$$P_X(k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, 2, \ldots, n \quad p \in [0, 1]$$

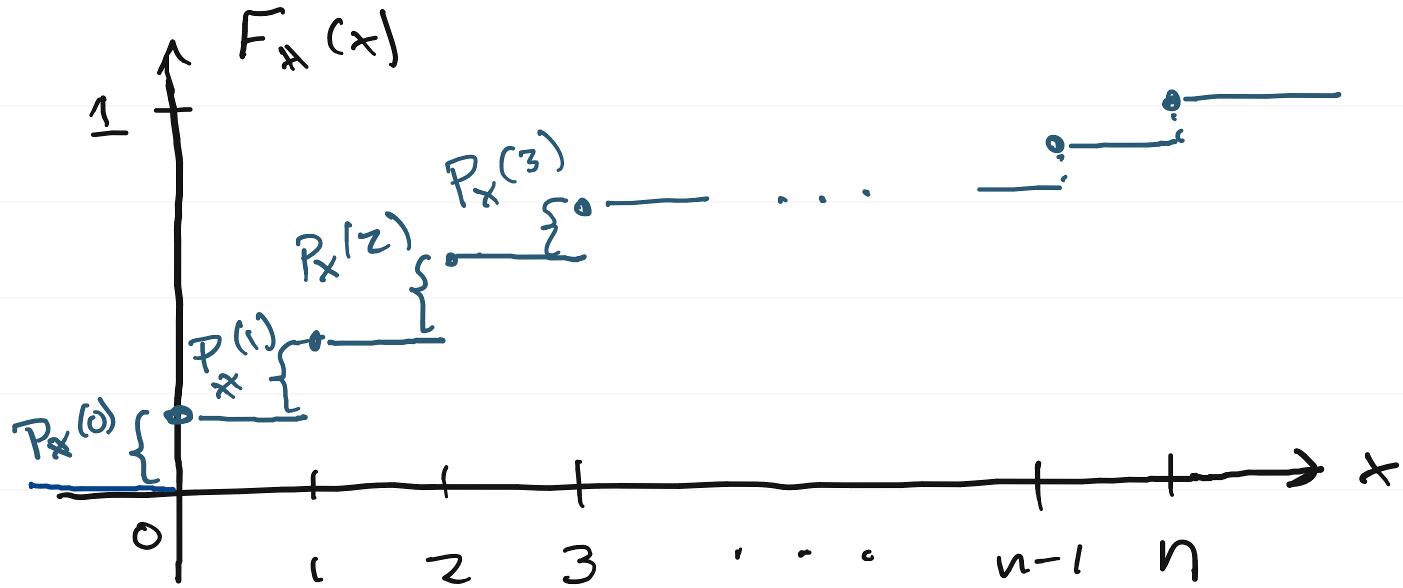
The cdf of this RV is

$$F_X(x) = P(\{X \leq x\}) = \sum_{k=0}^{m(x)} \binom{n}{k} p^k (1-p)^{n-k}$$

where $m(x)$ is an integer such that

$$m(x) \leq x < m(x) + 1.$$

13.4



$$f_X(x) = \text{“}\frac{dF_X(x)}{dx}\text{”} = \sum_{k=0}^{n} P_X(k)\,\delta(x-k)$$

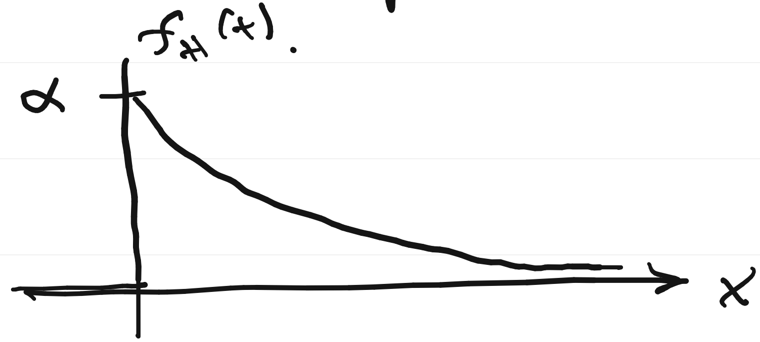
$$= \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k}\,\delta(x-k)$$

Ex. 4 Exponentially Distributed RV

A RV X with a pdf of the form

$$f_X(x) = \alpha e^{-\alpha x} \cdot 1_{[0,\infty)}(x) = \begin{cases} \alpha e^{-\alpha x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

where $\alpha > 0$ is called the parameter of the exponential RV.



See examples of other types of RVs in Papoulis

You should know the following pdfs and pmfs:

pmfs	pdfs
Binomial	Gaussian
Poisson	Exponential
Geometric	Uniform

Conditional Distributions

Given $(\mathcal{S}, \mathcal{F}, P)$ with a RV $\mathbb{X}$ defined on it, and given $A, M \in \mathcal{F}$, we know that

$$P(A|M) = \frac{P(A \cap M)}{P(M)}, \quad P(M) > 0.$$

Take $A = \text{"}\{\mathbb{X} \le x\}\text{"} = \{\omega \in \mathcal{S} : \mathbb{X}(\omega) \le x\}$

Then $P(A|M) = P(\{\mathbb{X} \le x\} | M)$

This is a conditional cdf of $\mathbb{X}$ conditioned on the event $M \in \mathcal{F}$.

Defn: The conditional cdf of the RV $\mathbb{X}$ conditioned on $M \in \mathcal{F}$ is

$$F_{\mathbb{X}}(x \mid M) = P(\{\mathbb{X} \le x\} \mid M)$$

$$= \frac{P(\{\mathbb{X} \le x\} \cap M)}{P(M)}, \quad x \in \mathbb{R}$$

The defn. of $F_{\mathbb{X}}(x|M)$ is just like the defn. of $F_{\mathbb{X}}(x)$, except we use the conditional prob. measure $P(\cdot|M)$ instead of $P(\cdot)$.

$P(\cdot \mid M)$ is a valid prob. measure $\Rightarrow$ $F_X(x \mid M)$ is a valid cdf

$\therefore$ $F_X(x \mid M)$ has all the properties of a valid cdf.

e.g. $P(\{a < X \leq b\} \mid M) = F_X(b \mid M) - F_X(a \mid M)$

Defn: The <u>conditional probability density function</u> (conditional pdf) of $\mathbb{X}$ conditioned on $M \in \mathcal{F}$ is

$$f_{\mathbb{X}}(x|M) \triangleq \frac{d F_{\mathbb{X}}(x|M)}{dx}.$$

n.b. $F_{\mathbb{X}}(x|M)$ is a valid cdf $\overset{\frac{d}{dx}}{\Longrightarrow}$ $f_{\mathbb{X}}(x|M)$ is a valid pdf

e.g. $P(\{a < \mathbb{X} \leq b\} | M) = \int_a^b f_{\mathbb{X}}(x|M)\, dx$

In general, we must know the underlying structure of $(\mathcal{S}, \mathcal{F}, P)$ and the mapping $\mathbb{X}$ to determine $F_{\mathbb{X}}(x|M)$ or $f_{\mathbb{X}}(x|M)$.

But sometimes we can describe the event $M \in \mathcal{F}$ in terms of the RV $\mathbb{X}$.

<u>e.g.</u>

1. $M = \{\mathbb{X} \leq a\}$, $a \in \mathbb{R}$.

2. $M = \{b < \mathbb{X} \leq a\}$, $a, b \in \mathbb{R}$
   $b <$

13.12

1. Let $M = \{ \mathbb{X} \leq a \}$, $a \in \mathbb{R}$

$$F_{\mathbb{X}}(x \mid M) = P(\{ \mathbb{X} \leq x \} \mid \{ \mathbb{X} \leq a \})$$

$$= \frac{P(\{ \mathbb{X} \leq x \} \cap \{ \mathbb{X} \leq a \})}{P(\{ \mathbb{X} \leq a \})}$$

Two cases: $x > a$ and $x \leq a$.

1. Let $M = \{\mathbf{X} \le a\}$

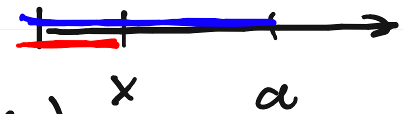
(a) $x > a$

$$\underline{\{\mathbf{X} \le x\}} \cap \underline{\{\mathbf{X} \le a\}} = \{\mathbf{X} \le a\}$$

$$F_{\mathbf{X}}(x \mid M) = \frac{P(\{\mathbf{X} \le a\})}{P(\{\mathbf{X} \le a\}} = \frac{F_{\mathbf{X}}(a)}{F_{\mathbf{X}}(a)} = 1$$

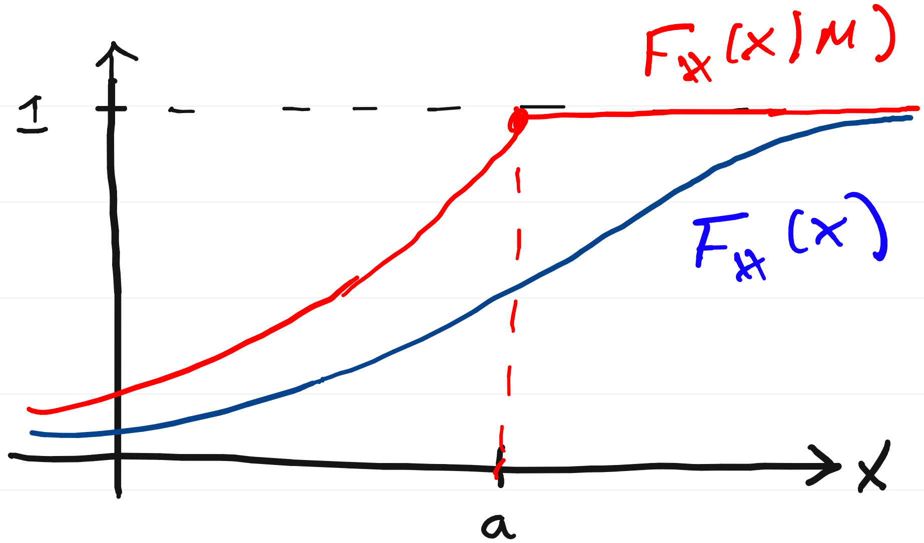
(b): **If $x \le a$:**

$$\{X \le x\} \cap \{X \le a\} = \{X \le x\}$$



$$F_X(x/M) = \frac{P(\{X \le x\})}{P(\{X \le a\})} = \frac{F_X(x)}{F_X(a)}$$

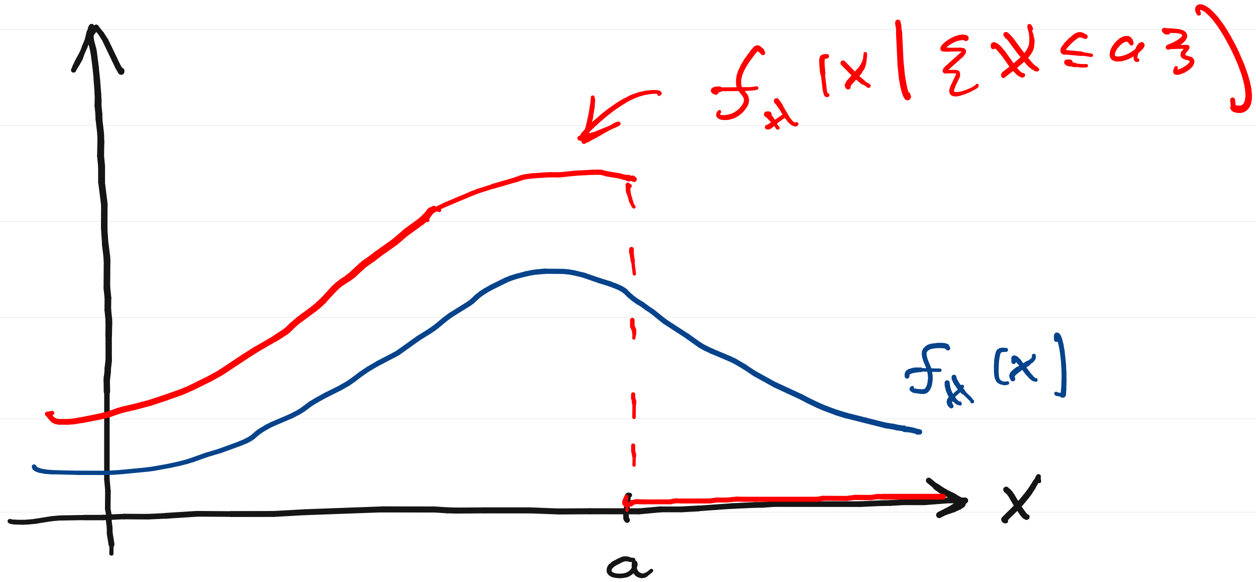
$$\therefore \quad F_X(x|M) = \begin{cases} \dfrac{F_X(x)}{F_X(a)}, & x \le a \\ 1, & x > a \end{cases}$$



Conditional pdf:

$$f_X(x|M) = \frac{dF_X(x|\{X \le a\})}{dx}$$

$$= \begin{cases} \dfrac{f_X(x)}{F_X(a)}, & x \le a \\ 0, & x > a \end{cases}$$


$f_X(x \mid \{X \le a\})$
$f_X(x)$
a
X

2. $M = \{b < \mathbb{X} \leq a\}$, $b < a$.

$$F_{\mathbb{X}}(x|M) = P(\{\mathbb{X} \leq x\} \mid \{b < \mathbb{X} \leq a\})$$

$$= \frac{P(\{\mathbb{X} \leq x\} \cap \{b < \mathbb{X} \leq a\})}{P(\{b < \mathbb{X} \leq a\})}$$

Three distinct regions:

(a) $x > a$

(b) $b < x \leq a$

(c) $x \leq b$

Analyzing these 3 cases, we get

(exercise)

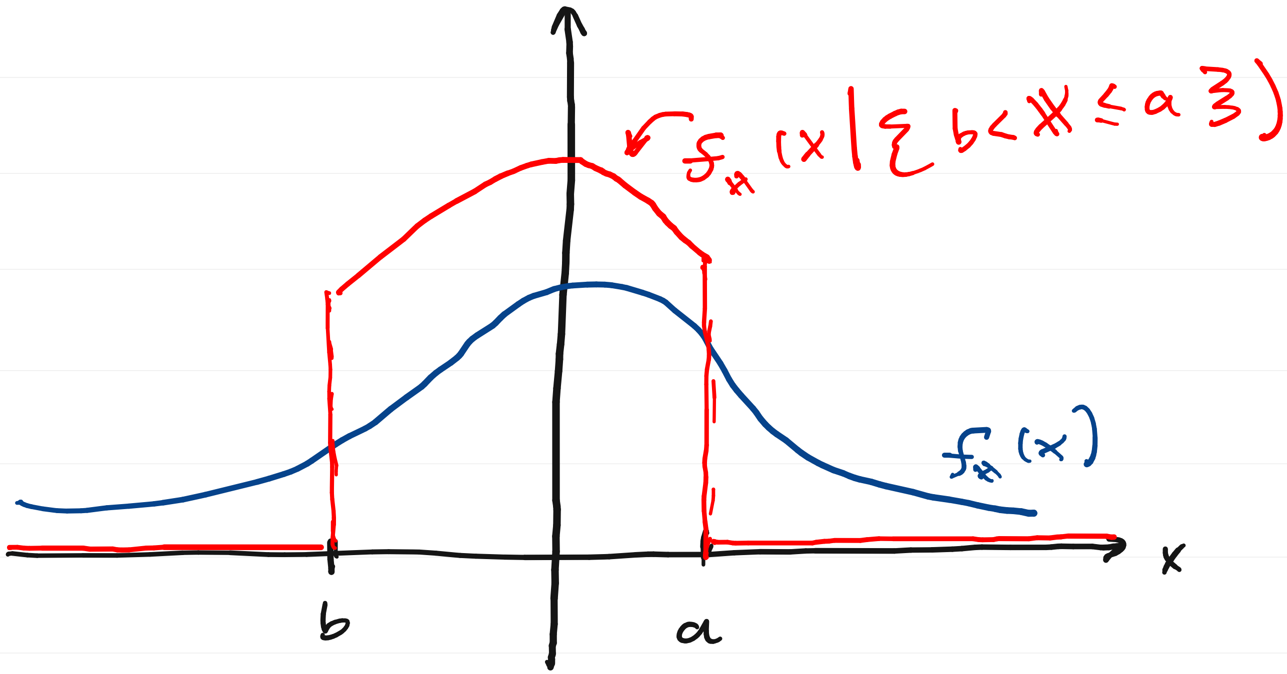
$$F_X(x|M) = \begin{cases} 1, & x > a \\ 0, & x \le b \\ \dfrac{F_X(x) - F_X(b)}{F_X(a) - F_X(b)}, & b < x \le a \end{cases}$$

The corresponding conditional pdf is

$$f_X(x|\{b < X \le a\}) = \frac{d F_X(x|\{b < X \le a\})}{dx}$$

$$= \begin{cases} 0, & x > a \\ 0, & x \le b \\ \dfrac{f_X(x)}{F_X(a) - F_X(b)}, & b < x \le a \end{cases}$$

A picture of these pdfs appears as follows:



The Total Prob. Law and Bayes Theorem

Given a RV $\mathbb{X}$ on $(\mathcal{S}, \mathcal{F}, P)$, let $\{A_1, \ldots, A_n\}$ be a partition of $\mathcal{S}$, with $A_k \in \mathcal{F}$, $k = 1, \ldots, n$.

Then

$$P(\{\mathbb{X} \le x\}) = P(\{\mathbb{X} \le x\} | A_1) P(A_1) + P(\{\mathbb{X} \le x\} | A_2) P(A_2) + \cdots + P(\{\mathbb{X} \le x\} | A_n) P(A_n) \quad \text{--- (1)}$$

But note that

$$P(\{\mathbb{X} \le x\}) = F_{\mathbb{X}}(x)$$

$$P(\{\mathbb{X} \le x\} | A_k) = F_{\mathbb{X}}(x | A_k)$$

$$\therefore \quad F_X(x) = F_X(x|A_1)P(A_1) + F_X(x|A_2)P(A_2) + \cdots + F_X(x|A_n)P(A_n) \quad \text{--- (1A)}$$

<u>note</u> $f_X(x) = \dfrac{dF_X(x)}{dx}$

and $f_X(x|A_k) = \dfrac{dF_X(x|A_k)}{dx}$

$$\Rightarrow \quad f_X(x) = f_X(x|A_1)P(A_1) + f_X(x|A_2)P(A_2) + \cdots + f_X(x|A_n)P(A_n) \quad \text{--- (1B)}$$

2. Recall Bayes Formula:

$$P(A|B) = \frac{P(B|A)\,P(A)}{P(B)}.$$

Let $B = \{X \le x\}$. Then

$$P(A|\{X \le x\}) = \frac{P(\{X \le x\}|A)\,P(A)}{P(\{X \le x\})}$$

$$= \frac{F_X(x|A)\,P(A)}{F_X(x)}$$

$$\therefore \quad P(A|\{X \le x\}) = \frac{F_X(x|A)\,P(A)}{F_X(x)}$$