ECE600: Random Variables and Waveforms

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Random Models in ECE

- Communications and Information Theory
- Computer Networks
- Solid State (Quantum Mechanics)
- Optics
- Control Theory
- Electromagnetics and Antennas
- Machine Learning, Big Data and Statistical Pattern ecognition

Probability is Used to Model Uncertainty

- **Systems that are too complex to model deterministically: (Ignorance)**
  - Maxwell: Theory of Gases
  - Boltzmann: Statistical Mechanics
- **Systems that are inherently random:**
  - Games of Chance
  - Quantum Mechanics
  - Other "fundamentally random" systems.

Set Theory

- Why Set Theory?
- A random experiment: Roll a fair die

$$S = \{1, 2, 3, 4, 5, 6\}$$

- We can define events:

$A_1 = \{1, 3, 5\}$ = outcome is odd

A_2 = outcome is divisible by 3 $= \{3, 6\}$

A_3 = outcome is prime $= \{2, 3, 5\}$

- Each event of interest is a subset of

$$\mathcal{S} = \{1, 2, 3, 4, 5, 6\}$$

- There are $2^6 = 64$ distinct subsets of $\mathcal{S}$.

<u>Events:</u>

- Events are subsets of $\mathcal{S}$.
- The collection of all events is called the event space:

$$\mathcal{F}(\mathcal{S}) = \{A_1, A_2, \ldots, A_{64}\}$$

Our random experiment is completely characterized by

$$\{ \mathcal{S} , \mathcal{F}(\mathcal{S}), P(\cdot) \}$$

where

$$P(\cdot) : \mathcal{F}(\mathcal{S}) \rightarrow [0,1]$$

and assigns probabilities to each event in $\mathcal{F}(\mathcal{S})$.

This framework – with minor modifications – will be used to describe all of the random experiments in this course.

A solid understanding of set theory will be important.

Basic Set Theory Definitions

- A set is simply a collection of objects.
  We intentionally leave this undefined.

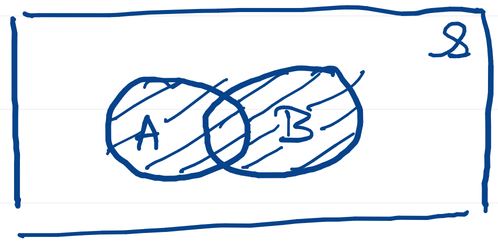
Defn: In any given set problem, the set containing all possible elements called the <u>universe</u>, the <u>universal set</u>, or the <u>space</u>. We typically denote it by $\mathcal{S}$.

<u>n.b</u> In probability the universal set is typically the sample space $\mathcal{S}$.

Set Operations:

Defn: The union of two sets A and B, denoted $A \cup B$, is defined as

$$A \cup B \triangleq \{\omega \in \mathcal{S} : \omega \in A \text{ or } \omega \in B\}$$

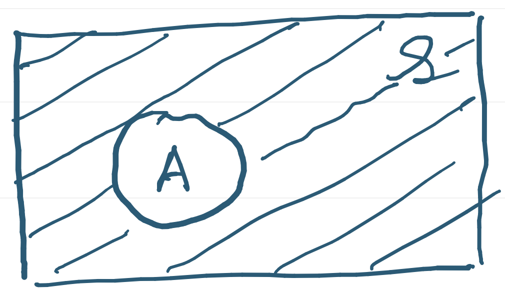


Defn: The intersection of two sets A and B, denoted $A \cap B$, is defined as

$$A \cap B \triangleq \{\omega \in \mathcal{S} : \omega \in A \text{ and } \omega \in B\}$$

Defn: The complement of a set A (with respect to $\mathcal{S}$), denoted $\bar{A}$, A' or A^c, is defined as

$$\bar{A} \triangleq \{ w \in \mathcal{S} : w \notin A \}$$



Defn: The empty set, denoted ϕ, contains no elements.

There are 3 fundamental set operations we have just defined:

Union: $A \cup B \triangleq \{\omega \in \mathcal{S} : \omega \in A \text{ or } \omega \in B\}$

Intersection: $A \cap B \triangleq \{\omega \in \mathcal{S} : \omega \in A \text{ and } \omega \in B\}$

Complement: $\bar{A} \triangleq \{\omega \in \mathcal{S} : \omega \notin A\}$

These are the three fundamental set operations, but there are two other "set difference operations" that are sometimes used:

Defn: The <u>set difference</u> of two sets A and B, denoted $A - B$ or $A \setminus B$, is defined as

$$A - B = \{\omega \in \Omega : \omega \in A \text{ and } \omega \notin B\}.$$

$$= A \cap \bar{B}$$

Defn: The symmetric difference between two sets A and B is defined as

$$A \,\Delta\, B = \{\omega \in \Omega : \omega \in A \text{ or } \omega \in B, \underline{\text{but not both}}\}.$$

$$= (A - B) \cup (B - A)$$

$$= (A \cup B) - (A \cap B)$$

$$= \cdots = (A \cap \bar{B}) \cup (\bar{A} \cap B)$$

<u>Defn</u>: Two sets A and B are <u>equal</u> if they contain exactly the same elements

<u>Fact</u>: Two sets A and B are equal if and only if $A \subset B$ and $B \subset A$.

<u>Proof</u>: Exercise

Algebra of Set Theory

1. $A \cup B = B \cup A.$ ($\cup$ is commutative)
2. $A \cap B = B \cap A.$ ($\cap$ is commutative)
3. $A \cup (B \cup C) = (A \cup B) \cup C.$ ($\cup$ is associative)
4. $A \cap (B \cap C) = (A \cap B) \cap C.$ ($\cap$ is associative)
5. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ ($\cap$ is distributive over union $\cup$).
6. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ ($\cup$ is distributive over $\cap$
7. $\overline{\overline{A}} = A$
8. $\overline{A \cap B} = \overline{A} \cup \overline{B}$ DeMorgan's Laws
9. $\overline{A \cup B} = \overline{A} \cap \overline{B}$
10. $\overline{\mathcal{S}} = \emptyset$
11. $A \cap \mathcal{S} = A$
12. $A \cap \emptyset = \emptyset$
13. $A \cup \mathcal{S} = S$ Obvious (?)
14. $A \cup \emptyset = A$
15. $A \cup \overline{A} = S$
16. $A \cap \overline{A} = \emptyset$

Defn: An <u>indexed collection of sets</u> is a set of sets

$$\{ A_i \; , i \in I \},$$

where I is an index set.

- So $\{ A_i \; ; i \in I \}$ is a "set of sets" or a "family of sets" or a "collection of sets."

Some Typical index Sets I:

$\mathbb{N} = \{1, 2, 3, \ldots\}$ = natural numbers.

$\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$ = non-negative integers

$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ = integers

$I_n = \{0, 1, 2, \ldots, n-1\}$

$\mathbb{R} = (-\infty, +\infty)$ = real line

Example: $I = \{1, 2, 3\}$

$$A_1 = [0,1] = \{x \in \mathbb{R} : 0 \le x \le 1\}$$
$$A_2 = [1,2]$$
$$A_3 = [2,3]$$

so $\{A_i ; i \in I\} = \{[0,1], [1,2], [2,3]\}$

Example: In our die rolling example,
$\mathcal{S} = \{1,2,3,4,5,6\}$.

we had $2^6 = 64$ possible subsets

$$\{A_i ; i \in I\} = \{A_1, A_2, \ldots, A_{64}\}$$

$$I = \{1, 2, 3, \ldots, 64\}$$