Homework Assignment #9
Should be completed by Session 28.

Reading Assignment: Read Sections 7-1 through 7-3 of Papoulis; or Sections 4-6, 4-8, and 4-9 of Shynk.

1. (Papoulis 6-78) Show that the jointly distributed random variables $X$ and $Y$ are statistically independent if and only if

$$E[U(a - X)U(b - Y)] = E[U(a - X)]E[U(b - Y)],$$

for all real numbers $a$ and $b$, where $U(\cdot) = 1_{[0,\infty)}(\cdot)$ is the unit step function.

2. (Papoulis 6-72) Show that if the random variables $X$ and $Y$ are independent and $Z = X + Y$, then $f_Z(z|x) = f_Y(z - x)$.

3. A signal $X$ consists of a zero-mean Gaussian random variable with variance $\sigma_X^2$. Noise $N$ consisting of an independent zero-mean Gaussian random variable with variance $\sigma_N^2$ is added to $X$ to produce the observation $Y = X + N$. Suppose we observe that $Y = y$.

   a) Find the minimum mean square error estimate $\hat{x}_{\text{MMS}}(y)$ of $X$.

   b) Find the MAP estimate $\hat{x}_{\text{MAP}}(y)$ of $X$.

4. Show that if constants $A$, $B$, and $a$ are such that

$$E[(Y - (AX + B))^2]$$

and

$$E[((Y - \eta_Y) - a(X - \eta_X))^2]$$

are minimum, then $a = A$.

5. Let $X$ and $Y$ be two jointly distributed Gaussian random variables having distribution $N(\eta_x, \eta_y, \sigma_x, \sigma_y, r_{xy})$ Let $V = aX + bY$ and $W = cX + dY$. Show that $V$ and $W$ are jointly Gaussian and find the parameters that characterize their joint density.

6. (Papoulis 7-2) The events $A$, $B$, and $C$ are such that

$$P(A) = P(B) = P(C) = 0.5$$

and

$$P(A \cap B) = P(A \cap C) = P(B \cap C) = P(A \cap B \cap C) = 0.25.$$ 

Show that the zero-one random variables associated with these events are not independent; they are however, independent in pairs.

7. (Papoulis 7-3) Show that if the random variables $X$, $Y$, and $Z$ are jointly Gaussian and independent in pairs, then they are independent.
8. (Papoulis 7-4) The random variables $X_i$ are i.i.d and uniformly distributed on the interval $(-0.5, 0.5)$. Show that

$$E[(X_1 + X_2 + X_3)^4] = \frac{13}{80}.$$  

(Hint: Use characteristic functions)

9. (Papoulis 7-7) Show that

$$E[X_1X_2|X_3] = E[E[X_1X_2|X_2X_3]|X_3] = E[X_2E[X_1|X_2X_3]|X_3].$$