EE600 Random Variables and Waveforms — Mark R. Bell
Spring 2024 — MSEE 336

Homework Assignment #9

Should be completed by Session 27.

Reading Assignment: Read Sections 7-1 through 7-3 of Papoulis.

1. (*Papoulis* 6-78) Show that the jointly distributed random variables $\mathbf{X}$ and $\mathbf{Y}$ are statistically independent if and only if

$$\mathrm{E}[U(a-\mathbf{X})U(b-\mathbf{Y})] = \mathrm{E}[U(a-\mathbf{X})]\mathrm{E}[U(b-\mathbf{Y})],$$

for all real numbers a and b, where $U(\cdot) = 1_{[0,\infty)}(\cdot)$ is the unit step function.

2. (*Papoulis* 6-72) Show that if the random variables $\mathbf{X}$ and $\mathbf{Y}$ are independent and $\mathbf{Z} = \mathbf{X} + \mathbf{Y}$, then $f_{\mathbf{Z}}(z|x) = f_{\mathbf{Y}}(z-x)$.

3. A signal $\mathbf{X}$ consists of a zero-mean Gaussian random variable with variance σ_X^2. Noise $\mathbf{N}$ consisting of an independent zero-mean Gaussian random variable with variance σ_N^2 is added to $\mathbf{X}$ to produce the observation $\mathbf{Y} = \mathbf{X} + \mathbf{N}$. Suppose we observe that $\mathbf{Y} = y$.
   (a) Find the minimum mean square error estimate $\hat{x}_{\mathrm{MMS}}(y)$ of $\mathbf{X}$.
   (b) Find the MAP estimate $\hat{x}_{\mathrm{MAP}}(y)$ of $\mathbf{X}$.

4. Show that if constants A, B, and a are such that

$$\mathrm{E}[(\mathbf{Y} - (A\mathbf{X} + B))^2]$$

and

$$\mathrm{E}[((\mathbf{Y} - \eta_Y) - a(\mathbf{X} - \eta_X))^2]$$

are minimum, then $a = A$.

5. Let $\mathbf{X}$ and $\mathbf{Y}$ be two jointly distributed Gaussian random variables having distribution $N(\eta_x, \eta_y, \sigma_x, \sigma_y, r_{xy})$ Let $\mathbf{V} = a\mathbf{X} + b\mathbf{Y}$ and $\mathbf{W} = c\mathbf{X} + d\mathbf{Y}$. Show that $\mathbf{V}$ and $\mathbf{W}$ are jointly Gaussian and find the parameters that characterize their joint density.

6. (*Papoulis* 7-2) The events A, B, and C are such that

$$P(A) = P(B) = P(C) = 0.5$$

and

$$P(A \cap B) = P(A \cap C) = P(B \cap C) = P(A \cap B \cap C) = 0.25.$$

Show that the zero-one random variables associated with these events are not independent; they are however, independent in pairs.

7. (*Papoulis* 7-3) Show that if the random variables $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ are jointly Gaussian and independent in pairs, then they are independent.

8. (*Papoulis* 7-4) The random variables $\mathbf{X}_i$ are i.i.d and uniformly distributed on the interval $(-0.5, 0.5)$. Show that

$$E[(\mathbf{X_1} + \mathbf{X_2} + \mathbf{X_3})^4] = \frac{13}{80}.$$

(*Hint:* Use characteristic functions)

9. (*Papoulis* 7-7) Show that

$$E[\mathbf{X_1}\mathbf{X}_2|\mathbf{X}_3] = E[E[\mathbf{X}_1\mathbf{X}_2|\mathbf{X}_2, \mathbf{X}_3]|\mathbf{X}_3] = E[\mathbf{X}_2 E[\mathbf{X}_1|\mathbf{X}_2, \mathbf{X}_3]|\mathbf{X}_3].$$