ECE600 Random Variables and Waveforms — Mark R. Bell
Spring 2024 — MSEE 336

Homework Assignment #8

Should be completed by Session 24

Reading Assignment: Read Sections 6-4 through 6-7 of Papoulis.

1. (*Papoulis* 6-40) Let $\mathbf{X}$ and $\mathbf{Y}$ be independent, discrete random variables having $P(\{\mathbf{X}=n\}) = a_n$ and $P(\{\mathbf{Y}=n\}) = b_n$, for $n = 0, 1, 2, \ldots$. Show that if $\mathbf{Z} = \mathbf{X}+\mathbf{Y}$, then
$$P(\{\mathbf{Z}=n\}) = \sum_{k=0}^{n} a_k b_{n-k}, \quad n = 0, 1, 2, \ldots .$$

2. *Papoulis* The random variables $\mathbf{X}$ and $\mathbf{Y}$ are Gaussian, independent, and have identical variance σ^2. Show that if
$$\mathbf{Z} = \sqrt{\mathbf{X}^2 + \mathbf{Y}^2},$$
then
$$f_Z(z) = \frac{z}{\sigma^2} I_0\left(\frac{z\eta}{\sigma^2}\right) e^{-(z^2+\eta^2)/2\sigma^2},$$
where $\eta = \sqrt{\eta_X^2 + \eta_Y^2}$, and $I_0(x)$ is the modified Bessel function of order zero:
$$I_0(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{x\cos\theta}\, d\theta.$$
Hint: See Example 6-16 on pages 191–192 of Papoulis.

3. (*Papoulis* 6-48) Show that if the random variables $\mathbf{X}$ and $\mathbf{Y}$ are jointly Gaussian and statistically independent, then
$$P(\{\mathbf{XY} < \mathbf{0}\}) = \boldsymbol{\Phi}\left(\frac{\eta_{\mathbf{X}}}{\sigma_{\mathbf{X}}}\right) + \boldsymbol{\Phi}\left(\frac{\eta_{\mathbf{Y}}}{\sigma_{\mathbf{Y}}}\right) - \mathbf{2}\boldsymbol{\Phi}\left(\frac{\eta_{\mathbf{X}}}{\sigma_{\mathbf{X}}}\right)\boldsymbol{\Phi}\left(\frac{\eta_{\mathbf{Y}}}{\sigma_{\mathbf{Y}}}\right),$$
where
$$\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx.$$

4. (*Papoulis* 6-49) Let $\mathbf{X}$ and $\mathbf{Y}$ be independent jointly distributed Gaussian random variables, both having mean 0 and variance σ^2. Show that if $\mathbf{Z} = |\mathbf{X}-\mathbf{Y}|$, then
$$\mathrm{E}[\mathbf{Z}] = \frac{2\sigma}{\sqrt{\pi}} \qquad \text{and} \qquad \mathrm{E}[\mathbf{Z^2}] = 2\sigma^2.$$

5. (*Papoulis* 6-50) Show that if $\mathbf{X}$ and $\mathbf{Y}$ are two jointly distributed independent exponential random variables, both with mean 1, and $\mathbf{Z} = (\mathbf{X}-\mathbf{Y}) \cdot 1_{[0,\infty)}(\mathbf{X}-\mathbf{Y})$, then $\mathrm{E}[\mathbf{Z}] = 1/2$.

6. (*Papoulis* 6-51) Show that for any two jointly distributed random variable $\mathbf{X}$ and $\mathbf{Y}$, real or complex,
   (a) $|\mathrm{E}[\mathbf{XY}]|^2 \leq \mathrm{E}[|\mathbf{X}|^2\}\mathrm{E}[|\mathbf{Y}|^2\}$; *(Schwarz inequality)*
   (b) $\sqrt{\mathrm{E}[|\mathbf{X}+\mathbf{Y}|^2} \leq \sqrt{\mathrm{E}[|\mathbf{X}|^2} + \sqrt{\mathrm{E}[|\mathbf{Y}|^2}$. *(triangle inequality)*
7. (*Papoulis* 6-52) Show that if correlation coefficient $r_{\mathbf{XY}}$ between two jointly distributed random variables equals 1, then $\mathbf{Y} = a\mathbf{X} + b$.
8. (*Papoulis* 6-54) Let $\mathbf{N}$ and $\mathbf{X}$ be two independent, jointly distributed random variables, where $\mathbf{N}$ is a Poisson random variable with mean λ and $\mathbf{X}$ has p.d.f.

$$f_{\mathbf{X}}(x) = \frac{\alpha}{\pi(\alpha^2 + x^2)}, \quad x \in (-\infty, \infty).$$

   Show that if $\mathbf{Z} = \mathbf{NX}$, then the characteristic function of $\mathbf{Z}$ is given by

$$\Phi_{\mathbf{Z}}(\omega) = \exp[\lambda e^{-\alpha|\omega|} - \lambda].$$

9. (*Papoulis* 6-71) Let $\mathbf{X}$ and $\mathbf{Y}$ be two independent, jointly distributed random variables that are both uniformly distributed on the interval $9-1,1)$. Find the conditional density $f_{\mathbf{R}}(r|M)$, of the random variable $\mathbf{R} = \sqrt{\mathbf{X}^2 + \mathbf{Y}^2}$, where M is the event $M = \{\mathbf{R} \leq 1\}$.
10. (*Papoulis* 6-74) We have a pile of m coins. The probability of "heads" on the i-th coin is p_i.We select one of the coins at random and toss it n times. "Heads"shows k times. Show that the probability we selected the r-th coin is

$$\frac{p_r^k(1-p_r)^{n-k}}{\sum_{i=1}^m p_i^k(1-p_i)^{n-k}}.$$

11. (*Papoulis* 6-77) Show that for any two jointly distributed random variable $\mathbf{X}$ and $\mathbf{Y}$, and any $\epsilon > 0$,

$$P(\{|\mathbf{X}-\mathbf{Y}| > \epsilon\}) \leq \frac{1}{\epsilon^2}\mathrm{E}[|\mathbf{X}-\mathbf{Y}|^2].$$