Homework Assignment #7
Should be completed by Session 21

Reading Assignment: Read Sections 6-4 through 6-7 of Papoulis or Sections 4.1 through 4.5 of Shynk.

1. If $X$ and $Y$ are zero-one random variables that indicate the events $A$ and $B$ respectively, i.e.,
   \[ X = 1_A(\omega) \]
   and
   \[ Y = 1_B(\omega), \]
   where $A, B \in \mathcal{F}$ of the probability space $(\mathcal{S}, \mathcal{F}, P)$, then
   (a) find the probability masses in the $(x, y)$-plane;
   (b) show that the random variables $X$ and $Y$ are statistically independent if and only if the events $A$ and $B$ are statistically independent.

2. (Papoulis 6-15) The random variables $X$ and $Y$ are statistically independent and $Y$ is uniformly distributed on the interval $(0, 1)$. Show that if
   \[ Z = X + Y, \]
   then
   \[ f_Z(z) = F_X(z) - F_X(z - 1). \]

3. (Papoulis 6-16) (a) The function $g(x)$ is monotone increasing and $Y = g(X)$. Show that
   \[ F_{XY}(x, y) = \begin{cases} F_X(x), & \text{if } y > g(x), \\ F_Y(y), & \text{if } y < g(x). \end{cases} \]
   (b) Find $F_{XY}(x, y)$ if $g(x)$ is monotone decreasing.

4. (Papoulis 6-24) Express $F_{ZW}(z, w)$ in terms of $F_{XY}(x, y)$ if $Z = \max(X, Y)$ and $W = \min(X, Y)$.

5. (Papoulis 6-18) The random variables $X$ and $Y$ are statistically independent with
   \[ f_X(x) = \frac{x}{\alpha^2} e^{-x^2/2\alpha^2} \cdot 1_{[0, \infty)}(x) \]
   and
   \[ f_Y(y) = \begin{cases} 1/\pi \sqrt{1 - y^2}, & \text{for } |y| \leq 1, \\ 0, & \text{elsewhere.} \end{cases} \]
   Show that the random variable $Z = XY$ is Gaussian with mean 0 and variance $\alpha^2$. 

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6. (Papoulis 6-19) The random variables $X$ and $Y$ are independent Rayleigh random variables with p.d.f.s

$$f_X(x) = \frac{x}{\alpha^2} e^{-x^2/(2\alpha^2)} \cdot 1_{[0,\infty)}(x) \quad \text{and} \quad f_Y(y) = \frac{y}{\beta^2} e^{-y^2/(2\beta^2)} \cdot 1_{[0,\infty)}(y).$$

(a) Show that if $Z = X/Y$, then

$$f_Z(z) = \frac{2\alpha^2}{\beta^2} \frac{z}{(z + \alpha^2/\beta^2)^2} \cdot 1_{[0,\infty)}(z).$$

(b) Using the result of part (a), show that for any $k > 0$,

$$P(\{X \leq kY\}) = \frac{k^2}{k^2 + \alpha^2/\beta^2}.$$

7. Consider two jointly distributed random variables $X$ and $Y$ having joint p.d.f.

$$f_{XY}(x, y) = \begin{cases} \frac{1}{\pi}, & \text{for } x + 2 + y^2 \leq 1; \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Find the marginal p.d.f. of $X$.

(b) Find the p.d.f. of the new random variable $R = \sqrt{X^2 + Y^2}$.

(c) Are $X$ and $Y$ statistically independent? Justify your answer (i.e., show whether or not they are independent.)

(d) Are $X$ and $Y$ uncorrelated? Justify your answer (i.e., show whether or not they are uncorrelated.)