ECE600 Random Variables and Waveforms
Spring 2024

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MSEE 336

Homework Assignment #6

Sould be completed by Session 17

Reading Assignment: Read Sections 5-4, 5-5, and 6-1 through 6-3 of Papoulis.

1. (*Papoulis* 5-22) (a) Show that if $\mathbf{Y} = a\mathbf{X} + b$, then $\sigma_{\mathbf{Y}} = |a|\sigma_{\mathbf{X}}$; (b) If $\mathbf{Y} = (\mathbf{X} - \eta_{\mathbf{X}})/\sigma_{\mathbf{X}}$, find $\eta_{\mathbf{Y}}$ and $\sigma_{\mathbf{Y}}$.(*n.b.* Here the symbol η refers to the mean and σ refers to the standard deviation.)
2. (*Papoulis* 5-23) Show that if $\mathbf{X}$ has a Rayleigh density with parameter α, such that

$$f_{\mathbf{X}}(x) = \frac{x}{\alpha^2} e^{-x^2/2\alpha^2} \cdot 1_{[0,\infty)}(x),$$

   and $\mathbf{Y} = b + c\mathbf{X}^2$, that $\sigma^2_{\mathbf{Y}} = 4c^2\alpha^4$.
3. (*Papoulis* 5-33) Showe that if $\mathbf{X}$ is a Gaussian random variable with mean η_x and variance σ^2, then

$$E[|\mathbf{X}|] = \sigma\sqrt{\frac{2}{\pi}} \cdot e^{-\eta_x^2/2\sigma^2} + 2\eta_x \Phi\left(\frac{\eta_x}{\sigma}\right) - \eta_x.$$

4. (*Papoulis* 5-27) Given a probability space $(\mathcal{S}, \mathcal{F}, P)$ and a partition $\{A_1, A_2, \ldots, A_n\}$ of $\mathcal{S}$, show that

$$E[\mathbf{X}] = E[\mathbf{X}|A_1]P(A_1) + E[\mathbf{X}|A_2]P(A_2) + \cdots + E[\mathbf{X}|A_n]P(A_n).$$

5. Let $\mathbf{X}$ be a binomially distributed random variable with

$$P(\{\mathbf{X} = k\}) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1. \ldots, n \ .$$

   Find the mean and variance of $\mathbf{X}$.
6. A Gamma distributed random variable $\mathbf{X}$ has pdf

$$f_X(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} 1_{(0,\infty)}(x), \quad \text{where} \ \ \Gamma(a) = \int_0^\infty u^{a-1} e^{-u}\, du,$$

   where $\alpha > 0$ and $\beta > 0$. Calculate the mean and variance of $\mathbf{X}$.
7. Compute the mean and variance of the uniformly distributed random variable $\mathbf{X}$ having pdf

$$f_X(x) = \frac{1}{b-a} 1_{[a,b]}(x).$$

8. (*Papoulis* 5-35) (*The Cheroff Bound*) (a) Show that for any $\alpha > 0$ and any $s \in \mathbf{R}$,

$$P(\{e^{s\mathbf{X}} \geq \alpha\}) \leq \frac{\phi_{\mathbf{X}}(s)}{\alpha},$$

where

$$\phi_{\mathbf{X}}(s) = E[e^{s\mathbf{X}}].$$

(*Hint: Use the Markov Inequality.*)
(b) Show that for any $A \in \mathbf{R}$,

$$P(\{\mathbf{X} \geq A\}) \leq e^{-sA}\phi_{\mathbf{X}}(s), \quad \forall s > 0,$$

$$P(\{\mathbf{X} \leq A\}) \leq e^{-sA}\phi_{\mathbf{X}}(s), \quad \forall s < 0.$$

(*Hint: Set* $\alpha = e^{sA}$ *in part (a).*)

9. (*Papoulis* 5-37) (a) Show that if $f(x)$ is a Cauchy density, *i.e.*

$$f(x) = \frac{\alpha}{\pi(\alpha^2 + x^2)}, \quad x \in \mathbf{R}, \quad \alpha > 0,$$

then the characteristic function $\Phi_{\mathbf{X}}(\omega) = e^{-\alpha|\omega|}$.
(b) Show that if $f(x)$ is a Laplace density, *i.e.*

$$f(x) = \frac{\alpha}{2}e^{-\alpha|x|}, \quad x \in \mathbf{R}, \quad \alpha > 0,$$

then the characteristic function of $\mathbf{X}$ is

$$\Phi_{\mathbf{X}}(\omega) = \frac{\alpha^2}{(\alpha^2 + \omega^2)}.$$

10. Consider a random variable $\mathbf{X}$ with characteristic function $\Phi_X(\omega)$.
(a) Show that if $\mathbf{Y} = a\mathbf{X} + b$, then $\Phi_Y(\omega) = e^{i\omega b}\Phi_X(a\omega)$.
(b) Show that if $\mathbf{X}$ is a Gaussian RV with mean 0 and variance 1, then

$$\Phi_X(\omega) = e^{-\omega^2/2}.$$

(c) Show that if $\mathbf{Y}$ is a Gaussian RV with mean μ and variance σ^2, then

$$\Phi_Y(\omega) = e^{i\omega\mu - \omega^2\sigma^2/2}.$$