ECE600 Random Variables and Waveforms Spring 2024

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MSEE 336

Homework Assignment #5
Should be completed by Session 15

Reading Assignment: Reread Sections 5-1 and 5-2, and read Sections 5-3 through 5-5 of Papoulis.

1. (*Papoulis* 5-2) Find $F_{\mathbf{Y}}(y)$ and $f_{\mathbf{Y}}(y)$ if $\mathbf{Y} = -4\mathbf{X} + 3$ and $\mathbf{X}$ is an exponentially distributed random variable with p.d.f. $F_{\mathbf{X}}(x) = 2e^{-2x} \cdot 1_{[0,\infty)}(x)$.
2. (*Papoulis* 5-3) If the random variable $\mathbf{X} \sim \mathcal{N}(0, c^2)$ and
$$g(x) = \begin{cases} x - c, & \text{for } x \geq c, \\ 0, & \text{for } -c < x < c, \\ x + c, & \text{for } x \leq -c, \end{cases}$$
Find and sletch $f_{\mathbf{Y}}(y)$ and $F_{\mathbf{Y}}(y)$ if $\mathbf{Y} = g(\mathbf{X})$.
3. (*Papoulis* 5-4) If $\mathbf{X}$ is a uniformly distributed random variable on the interval $(-2c, 2c)$, where $c > 0$, and $\mathbf{Y} = \mathbf{X}^2$, find and sketch $f_{\mathbf{Y}}(y)$ and $F_{\mathbf{Y}}(y)$.
4. (*Papoulis* 5-7(a)) We place 200 points at random in the interval $(0, 100)$. The distance from 0 to the smallest of the 100 points is the random variable $\mathbf{Z}$. Find $F_{\mathbf{Z}}(z)$.
5. (*Papoulis* 5-9) Express the density $f_{\mathbf{Y}}(y)$ of the random variable $\mathbf{Y} = g(\mathbf{X})$ in terms of $f_{\mathbf{X}}(x)$ if $\mathbf{Y} = g(\mathbf{X})$ when (a) $g(x) = |x|$, and (b) $g(x) = e^{-x} \cdot 1_{[0,\infty)}(x)$.
6. (*Papoulis* 5-11.) Show that if the random variable $\mathbf{X}$ has a Cauchy p.d.f.
$$f_{\mathbf{X}}(x) = \frac{1}{\pi(1 + x^2)}, \quad -\infty < x < \infty,$$
and $\mathbf{Y} = \arctan(\mathbf{X})$, then $\mathbf{Y}$ is uniformly distributed on the interval $(-\pi/2, \pi/2)$.
7. Let $\mathbf{X}$ be a Gaussian random variable with mean $\mu = 1$ and standard deviation $\sigma = 1$. Let $g_1(x)$ be the function shown in Fig. 1 below. Define a new random variable $\mathbf{Y} = g_1(\mathbf{X})$. Find expressions for the cdf $F_Y(y)$ and the pdf $f_Y(y)$, and sketch them.
8. Let $\mathbf{X}$ be an exponential random variable with pdf
$$f_X(x) = e^{-x} \cdot 1_{[0,\infty)}(x).$$
Let $g_2(x)$ be the function shown in Fig. 2 below. Define a new random variable $\mathbf{Y} = g_2(\mathbf{X})$. Find expressions for the cdf $F_Y(y)$ and the pdf $f_Y(y)$, and sketch them.

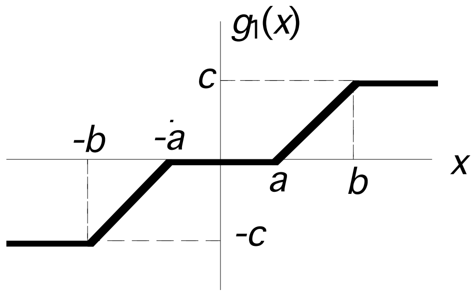

Figure 1

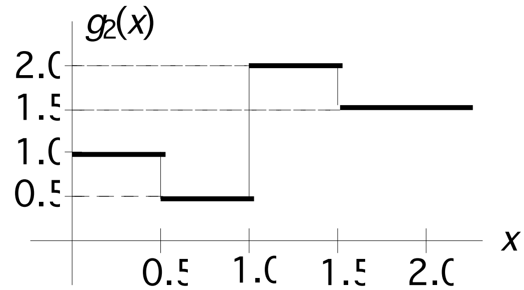

Figure 2

9. Let $\mathcal{S} = \mathbf{R}$, and let $A = [1, 3] \subset \mathcal{S}$.
   (a) Show that the four sets $\{\emptyset, A, \overline{A}, \mathcal{S}\}$ constitute a σ-field.
   (b) Define the function $\mathbf{X}$ on $\mathcal{S}$ by $\mathbf{X}(\omega) = \omega^2$. Show that $\mathbf{X}$ is *not* a random variable if the four sets specified in A are the only events in the event space. (*Hint: Pick any interval* $(a, b) \subset \mathbf{R}$ *and show that* $\{\omega : \mathbf{X}(\omega) \in (a, b)\}$ *is not an event.*)
   (c) Define the function $\mathbf{Y}(\omega) = 2 \cdot 1_A(\omega) + 3 \cdot 1_{\overline{A}}(\omega)$. Show that $\mathbf{Y}(\omega)$ is a random variable. (*Hint: Show that for any interval* $(a, b) \subset \mathbf{R}$ *that you pick,* $\{\omega : \mathbf{X}(\omega) \in (a, b)\}$ *is one of the four events in part (a).*)