ECE600 Random Variables and Waveforms
Spring 2024

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MSEE 336

Homework Assignment #4
Should be completed by Session 13

Reading Assignment: All of Chapter 4, and Sections 5-1 through 5-2 of Papoulis.

1. (*Papoulis* 4-1) Suppose that x_u is the u percentile of the random variable $\mathbf{X}$, that is, $F_{\mathbf{X}}(x_u) = u$. Show that if $f_{\mathbf{X}}(-x) = f_{\mathbf{X}}(x), \forall x \in \mathbf{R}$, then $x_{1-u} = -x_u$.
2. (*Papoulis* 4-2) Show that if $f_{\mathbf{X}}$ is symmetrical about the point $x = \eta$ and
$$P(\{\eta - a < \mathbf{X} < \eta + a\}) = 1 - \alpha,$$
then $a = \eta - x_{\alpha/2} = x_{1-\alpha/2} - \eta$. (*n.b.,* α and a are different variables.)
3. (*Papoulis* 4-11) Assume the sample space $\mathcal{S}$ consists of all points t_i in the interval $(0, 1)$ and $P(\{0 \leq t_i \leq y\}) = y$ for every $y \in [0, 1]$. The function $G(x)$ is increasing from $G(-\infty) = 0$ to $G(\infty) = 1$; hence $G(\cdot)$ has an inverse $G^{-1}(y) = H(y)$. The random variable $\mathbf{X}$ is defined as $\mathbf{X}(t_i) = H(t_i)$. Show that $F_{\mathbf{X}}(x) = G(x)$.
4. Consider the result of the previous problem. Now suppose you have a random number generator that generates random variables with pdf
$$f_X(x) = 1_{(0,1)}(x)$$
and suppose you want to generate a random variable Z with pdf $f_Z(z)$. How would you process the output of the uniform random number generator, X, to do this?
5. (*Papoulis* 4-13) A fair coin is tossed three times and the random variable $\mathbf{X}$ equals the total number of "Heads" that occur in the three tosses. Find and sketch $F_{\mathbf{X}}(x)$ and $f_{\mathbf{X}}(x)$.
6. (*Papoulis* 4-16) Let ξ is the outcome of a random experiment and two random variables $\mathbf{X}$ and $\mathbf{Y}$ defined on the random experiment. Show that if $\mathbf{X}(\xi) \leq \mathbf{Y}(\xi), \forall \xi \in \mathcal{S}$, then
$$F_{\mathbf{X}}(w) \geq F_{\mathbf{Y}}(w), \quad \forall w \in \mathbf{R}.$$
7. (*Papoulis* 4-17) Show that if
$$\beta(t) = f_{\mathbf{X}}(t|\{\mathbf{X} > t\})$$
is the *conditional failure rate* of the random variable $\mathbf{X}$ and $\beta(t) = kt$ for a positive constant k, then $f_{\mathbf{X}}(x)$ is a Rayleigh density.
8. (*Papoulis* 4-19) Show that
$$F_{\mathbf{X}}(x|A) = \frac{P(A|\{\mathbf{X} \leq x\})F_{\mathbf{X}}(x)}{P(A)}.$$

9. (*Papoulis* 4-21) The probability of *heads* of a random coin is a random variable $\mathbf{p}$ uniformly distruted on the unit interval $(0, 1)$.
   (a) Find $P(\{0.3 \le \mathbf{p} \le 0.7\})$.
   (b) The coin is tossed 10 times and *heads* shows 6 times. Find the *a posteriori* probsbility that $\mathbf{p}$ os between 0.3 and 0.7.
10. Show that the Gaussian pdf

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{\frac{-(x-\mu)^2}{2\sigma^2}\right\}$$

satisfies the condition

$$I = \int_{-\infty}^{\infty} f_X(x)\, dx = 1.$$

Hint: It might be easier to find I^2 and then determine I.

11. Let X have exponential distribution

$$f_X(x) = \frac{1}{\mu} e^{-x/\mu} 1_{[0,\infty)}(x).$$

Find the conditional density $f_X(x|\mu < X \le 2\mu)$.