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MSEE 336

Homework Assignment #4
Due Monday, October 21, 2024

Reading Assignment: Read the Bayesian Detection section in Chapter 5 and Chapter 6 on Matched Filters of the course notes.

1. Consider the following detection problem defined in the Bayesian context:
   (a) Call $f(\underline{\theta}, \underline{x})$ the joint probability mass function-probability density function of $\underline{\theta}$ and $\underline{x}$:
$$f(\underline{\theta}, \underline{x}) = \begin{cases} p_0 f_{\underline{\theta}_0}(\underline{x}), & \underline{\theta} = \underline{\theta}_0; \\ p_1 f_{\underline{\theta}_1}(\underline{x}), & \underline{\theta} = \underline{\theta}_1. \end{cases}$$
   Find the conditional probability mass function of $\underline{\theta}$ given $\underline{X}$.
   (b) One criterion for testing H_0 versus H_1 would be to evaluate the posterior conditioal pmf (part (a)) at $\underline{\theta}$ for the measurement $\underline{X}$—call the result the *posterior likelihood*— and select H_1 when the posterior likelihood of H_1 exceeds the posterior likelihood of H_0. Show that the resulting test is a likelihood ratio test, and find the threshold to which the likelihood ratio $L(\underline{X})$ must be compared.

2. *(Binary Optical Communication):* A laser transmits a binary digit ("0" or "1") by transmitting N_T photons in the time interval $[0, T]$, where the number of photons transmitted is a Poisson random variable with pmf
$$p_i(n) = P(\{N_T = n\}) = \frac{(\lambda_i T)^n e^{-\lambda_i T}}{n!}, \quad n = 0, 1, 2, \ldots, \quad i = 0, 1.$$
   Here the rate parameter λ_i is λ_0 under H_0 and λ_1 under H_1, where $\lambda_1 > \lambda_0 \geq 0$. Now assume that each transmitted photon has a probability p of being detected. Furthurmore, assume that the digits "0" and "1" have prior probabilities p_0 and p_1, respectively. You are to decide whether a "0" or "1" was sent based on the observed number of *detected* photons M_T in the interval $[0, T]$.
   (a) Find the probability mass function of M_T.
   (b) Find the decision rule that minimizes the probability of error, and express it as a function of λ_0, λ_1, p, p_0, p_1 and T.
   (c) Write an expression for the resulting probability of error.

3. Assume that H_0 is the hypothesis that no target is present and H_1 is the hypothesis that a target is present. The prior probability of H_1 is p_1. Assume that the loss associated with a type I error is L_{01} and with a type II error is L_{10}; also assume that $L_{00} = L_{11} = 0$. We observe a scalar random variable X, which under H_0 has pdf
$$f_0(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{\frac{-x^2}{2\sigma^2}\right\},$$

and under H_1 has pdf

$$f_1(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{\frac{-(x-A)^2}{2\sigma^2}\right\},$$

where A is a positive constant.

(a) Find a threshold test on X yielding the Bayes decision rule, and specify the threshold x_0 to which X must be compared.

(b) Compute the value for the threshold x_0 as a funtion of p_1 when $A = 1$, $\sigma = 1$, $L_{01} = 1$, and $L_{10} = 2$.

(c) Compute α, the probability of type I error, and $1 - \beta$, the probability of type II error, as a function of p_1 for the numerical values given in part (b). Plot these for $0 < p_1 < 0.9$.

(d) Compute the Bayes risk for the values given in part (b) and plot it for $0.1 < p_1 < 0.9$.

(e) The minimax test is the test that minimizes the Bayes risk over all possible priors p_1. Consider the result of part (d), and find the minimanx test corresponding to the values given in part (b).

4. Consider the signal

$$s(t) = A \cdot 1_{[0,T]}(t)$$

in additive white Gaussian noise with power spectral density

$$S_{nn}(f) = N_o/2,$$

where A is a positive constant and T is the duration of the pulse.

(a) Find the impulse response of the matched filter whose output is sampled at time T.

(b) Find the peak SNR at the output of the matched filter in this case.

5. Consider the signal

$$s(t) = Ae^{i\pi\alpha t^2} \cdot 1_{[0,T]}(t)$$

in additive white Gaussian noise with power spectral density

$$S_{nn}(f) = N_o/2,$$

where A is a positive constant, T is he duration of the pulse, and $\alpha > 0$ is the "chirp rate" of this linear FM pulse.

(a) Find the impulse response of the matched filter whose output is sampled at time T.

(b) Find the peak SNR at the output of the matched filter in this case.