ECE678 Radar Engineering
Fall 2024

Mark R. Bell
MSEE 336

Homework Assignment #3
Due Friday, October 11, 2024

Reading Assignment: Bell and Chang, Chapter 5 "Detection Theory", (Ref: Scharf, Sections 4.1–4.3 and 4.7).

1. Let X be a random variable. Under hypothesis H_0, X comes from an exponential distribution with mean μ_0:

$$f_0(x) = \frac{1}{\mu_0} e^{-x/\mu_0} \cdot 1_{[0,\infty)}(x).$$

Under hypothesis H_1, X comes from an exponential distribution with mean μ_1:

$$f_1(x) = \frac{1}{\mu_1} e^{-x/\mu_1} \cdot 1_{[0,\infty)}(x).$$

Assume that $\mu_1 > \mu_0$.

(a) Find the most powerful test of size α for testing hypothesis H_0 against the alternative H_1.

(b) Find an expression for the power β of this test in terms of the size α and the parameters μ_0 and μ_1.

(c) Plot the ROC of this statistical test for the cases of $\mu_1/\mu_0 = 2,\ 5,\ 10$. Plot these three cases on the same set of axes.

2. Let $\mathbf{X} = (X_1, \ldots, X_N)^T$ be an independent identically distributed random sample of N random variables. Under hypothesis H_0, each of the random variables comes from an exponential distribution with mean μ_0:

$$f_0(x) = \frac{1}{\mu_0} e^{-x/\mu_0} \cdot 1_{[0,\infty)}(x).$$

Under hypothesis H_1, each of the random variables comes from an exponential distribution with mean μ_1:

$$f_1(x) = \frac{1}{\mu_1} e^{-x/\mu_1} \cdot 1_{[0,\infty)}(x).$$

Assume that $\mu_1 > \mu_0$.

(a) Find the most powerful test of size α for testing hypothesis H_0 against the alternative H_1.

(b) Find an expression for the power β of this test in terms of the size α and the parameters μ_0 and μ_1.

(c) Plot the ROC of this statistical test for $\mu_1/\mu_0 = 2$ and $N = 1,\ 2,\ 10$. Plot these three cases on the same set of axes.

Note: the sum of N i.i.d exponentially distributed random variables is a chi-square random variable with $2N$ degrees of freedom when properly normalized.

3. In a laser radar system, the number of photons that strike a photon-counter detector during an observation interval is a Poisson random variable N, with mean $\lambda_1 = 5$ when a target is present and $\lambda_0 = 1$ when no target is present. Construct the most powerful test of the hypothesis H_1 that a target is present versus H_0 that no target is present having a probability of false alarm of exactly 0.01. What is the probability of detection for this test?
4. Let $\mathbf{X}$ be a Gaussian random vector with mean $\mathbf{m}_0$ and covariance C_0 under hypothesis H_0, and mean $\mathbf{m}_1$ and covariance C_1 under hypothesis H_1. Find the form of the likelihood ratio test of H_0 versus H_1 based on an observation of $\mathbf{X}$.
5. Assume that X is an exponentially distributed random variable with mean $\mu \in \Theta = [0, \infty)$. Assume that the compound hypothesis H_0 has the parameter set $\Theta_0 = (0, 1]$ associated with it and the compound hypothesis H_1 has the parameter set $\Theta_1 = (1, \infty)$. Construct the uniformly most powerful test of H_0 versus H_1 of size α. Plot the power $\beta(\mu)$ of this test for $\mu \in \Theta_1$ for this test.
6. The output of one common type of radar receiver has the property that the *amplitude* of its output sampled at any point in time is a Rayleigh distributed random variable R with pdf

$$f_R(r) = \frac{r}{\sigma^2} e^{-r^2/2\sigma^2} \cdot 1_{[0,\infty)}(r),$$

when zero-mean Gaussian noise only is present at its input, and is a Rician random variable with pdf

$$f_R(r) = \frac{r}{\sigma^2} \exp\left\{ \frac{-(r^2 + A^2)}{2\sigma^2} \right\} I_0\left(\frac{rA}{\sigma^2} \right) \cdot 1_{[0,\infty)}(r),$$

when Gaussian noise plus a constant target signal of amplitude A is present at its input. Here σ^2 is the variance of the Gaussian receiver noise, A is the amplitude of the constant target return, and $I_0(\cdot)$ is the modified zero-order Bessel function of the third kind:

$$I_0(x) = \frac{1}{2\pi} \int_0^{2\pi} \exp\left[-x \cos \phi\right] d\phi.$$

   (a) Write down the likelihood ratio test to test the hypothesis H_0 that no target is present versus H_1 that the target H_1 is present.
   (b) What is the threshold on R to achieve a test of size α.
   (c) Write down the integral expression for the power of the test.
   (d) Using the *Marcum-Q Function*, defined as

$$Q(a, b) = \int_b^{\infty} x \exp\left\{ \frac{-(x^2 + a^2)}{2} \right\} I_0(ax)\, dx,$$

   express the power of the test in terms of the Marcum-Q function.