Session 28

Recall...

28.1

Defn: A random sequence $\{X_n\}$ <u>converges in distribution</u> (d) to a RV X if

$$F_{X_n}(x) \longrightarrow F_X(x) \quad \text{as } n \to \infty \quad (*)$$

at every point $x \in \mathbb{R}$ where $F_X(x)$ is continuous.

(*): i.e., $\forall \varepsilon > 0$, $\exists n_\varepsilon \in \mathbb{N}$ such that

$$|F_{X_n}(x) - F_X(x)| < \varepsilon, \quad \forall n \geq n_\varepsilon$$

for all $x \in \mathbb{R}$ where $F_X(x)$ is continuous.

Recall...

28.2

Defn: A random sequence $\{X_n\}$ converges in density (den) to a R.V. X if

$$f_{X_n}(x) \longrightarrow f_X(x) \text{ as } n \to \infty$$

at every point $x \in \mathbb{R}$ where $F_X(x)$ is continuous.

28.3

Aren't convergence in density and convergence in distribution equivalent?

No!

Example: Let $\{X_n\}$ be a sequence of RVs having p.d.fs

$$f_{X_n}(x) = \left[1 + \cos(2\pi n x)\right] \cdot 1_{[0,1]}(x).$$

n.b. (i) $f_{X_n}(x) \geq 0, \ \forall x \in \mathbb{R}$

(ii) $\int_{-\infty}^{\infty} f_{X_n}(x)\,dx = \int_0^1 (1+\cos(2\pi n x))\,dx = \left(x + \frac{1}{2\pi n}\sin(2\pi n x)\right)\Big|_0^1$

$= 1, \quad n = 1, 2, 3, \ldots$

28.4

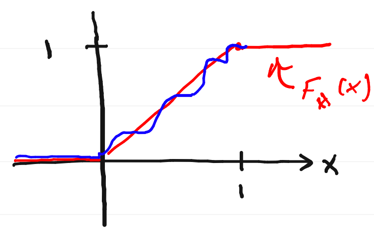
Now define

$$F_X(x) = \begin{cases} 0, & x < 0, \\ x, & x \in [0,1], \\ 1, & x > 1. \end{cases}$$

Note that

$$F_{X_n}(x) = \int_{-\infty}^{x} f_n(\alpha)\, d\alpha = \begin{cases} 0, & x < 0 \\ x + \frac{1}{2\pi n}\sin(2\pi n x), & x \in [0,1] \\ 1, & x > 1 \end{cases}$$

Clearly $F_{X_n}(x) \to F_X(x)$, $\forall x \in \mathbb{R}$



$F_{X_n}(x) \to F_X(x)$ as $n \to \infty$

$\therefore$ We have convergence in distribution

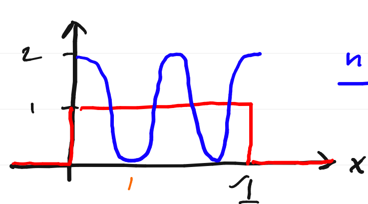
28.5

But $f_{X_n}(x) \not\to f_X(x)$

$$f_X(x) = \frac{d F_X(x)}{dx} = \mathbb{1}_{[0,1]}(x)$$

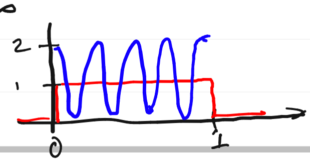
and

$$f_n(x) = [1 + \cos 2\pi n x] \cdot \mathbb{1}_{[0,1]}(x)$$



$n = 2$

As $n \to \infty$



$f_{X_n}(x) \not\to f_X(x)$ as $n \to \infty$.

$\therefore$ Convergence (d) $\not\Rightarrow$ convergence (den)

28.6

However

$$\text{convergence (den)} \not\Rightarrow \text{convergence (d)}.$$

(n.b. The integration that converts $f_{\#_n}(x)$ into $F_{\#_n}(x)$ smoothes things out.)

28.7

Cauchy Criterion for Convergence

Recall that a sequence of real numbers $x_1, \dots, x_n, \dots$ converges to a limit x if $\forall \varepsilon > 0$, $\exists n_\varepsilon \in \mathbb{N}$ such that

$$|x_n - x| < \varepsilon, \quad \forall n \geq n_\varepsilon.$$

To use this definition to determine if $x_1, \dots, x_n, \dots$ converges, we have to know the limit x.

28.8

The Cauchy Criterion gives us a way to test for convergence without knowing the limit x.

Cauchy Criterion: If $\{x_n\}$ is a sequence of real numbers and

$$|x_{n+m} - x_n| \to 0 \quad \text{as} \quad n \to \infty$$

for all $m \in \mathbb{N}$, then the sequence converges to a real number.

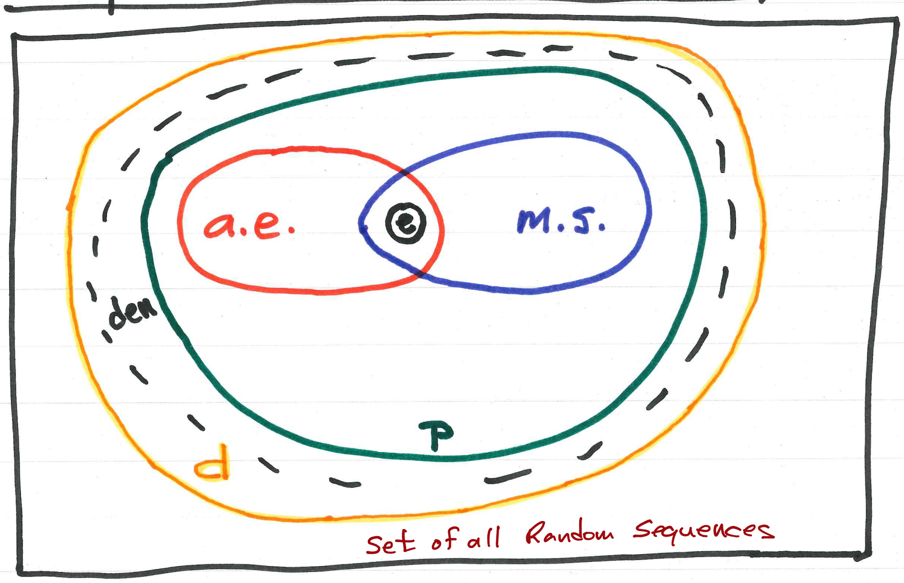
n.b. The Cauchy criterion can be applied to various forms of stochastic convergence.

e.g. $E\left[|X_{n+m} - X_n|^2\right] \to 0$ as $n \to \infty$ for all $m = 1, 2, 3, \ldots$

Then $\{X_n\}$ converges in mean-square.

28.9

Comparison of Modes of Convergence



26.21

1. M.S. convergence $\Rightarrow$ convergence (p) 28.10

n.b. $P(\{|X-\mu| \geq \varepsilon\}) \leq \frac{E[|X-\mu|^2]}{\varepsilon^2} = \frac{Var(X)}{\varepsilon^2}$

$$\Rightarrow P(\{|X_n - X| \geq \varepsilon\}) \leq \frac{E[|X_n - X|^2]}{\varepsilon^2}$$

$E[|X_n - X|^2] \xrightarrow[n\to\infty]{} 0$ (m.s convergence)

$\Rightarrow P(\{|X_n - X| \geq \varepsilon\}) \to 0$ as $n \to \infty$.

(convergence (p))

2. convergence (a.e.) $\Rightarrow$ convergence (p). 28.11

Follow from definitions

Converse is not true.

(See Papoulis).

3. Convergence (d) is "weaker" than convergence (a.e.), (m.s.) or (p).

(a.e.) $\Rightarrow$ (d)

(m.s) $\Rightarrow$ (d)

(p) $\Rightarrow$ (d).

4. $(a.e) \not\Rightarrow (m.s)$

$(m.s) \not\Rightarrow (a.e.)$

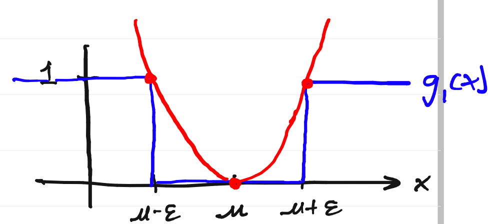
<u>n.b.</u> The Chebyshev Inequality is an important tool when working with (m.s.) convergence.

<u>Chebyshev Inequality:</u> Let $\mathbb{X}$ be a R.V. with mean μ and variance $\sigma^2 < \infty$. Then $\forall \varepsilon > 0$,

$$P(\{|\mathbb{X}-\mu| \geq \varepsilon\}) \leq \frac{\sigma^2}{\varepsilon^2}$$

<u>Proof:</u> Let $g_1(x) = \mathbb{1}_{\{r \in \mathbb{R} : |r-\mu| \geq \varepsilon\}}(x)$

and $g_2(x) = \dfrac{(x-\mu)^2}{\varepsilon^2}$



Note: $g_2(x) \geq g_1(x), \ \forall x \in \mathbb{R}$

Let $\phi(x) = g_2(x) - g_1(x) \geq 0, \ \forall x \in \mathbb{R}$

$\Rightarrow E[\phi(\mathbb{X})] \geq 0$

$\Rightarrow E[\phi(\mathbb{X})] = E[g_2(\mathbb{X}) - g_1(\mathbb{X})] = E[g_2(\mathbb{X})] - E[g_1(\mathbb{X})] \geq 0$

$\Rightarrow E[g_1(\mathbb{X})] \leq E[g_2(\mathbb{X})]$

But $E[g_1(\mathbb{X})] = P(\{|\mathbb{X}-\mu| \geq \varepsilon\})$

$E[g_2(\mathbb{X})] = \dfrac{\sigma^2}{\varepsilon^2}$

$\therefore P(\{|\mathbb{X}-\mu| \geq \varepsilon\}) \leq \dfrac{\sigma^2}{\varepsilon^2}$ ∎

Recall ...

The Weak of Large Numbers

28.14

Let $\{X_n\}$ be a sequence of i.i.d Random variables with mean μ and variance σ^2.

Define
$$\Psi_n = \frac{1}{n}\sum_{k=1}^{n} X_k \quad , n=1,2,3,\ldots$$

Then for any $\varepsilon > 0$,
$$P(\{|\Psi_n - \mu| \geq \varepsilon\}) \longrightarrow 0 \quad \text{as } n \to \infty.$$
$$(\Psi_n \xrightarrow{(p)} \mu).$$

Recall...

28.15

Proof: $E[\Psi_n] = E\left[\frac{1}{n}\sum_{k=1}^{n} X_k\right]$
$$= \frac{1}{n}\sum_{k=1}^{n} E[X_k] = \frac{1}{n}(n\mu) = \boxed{\mu}$$

and $\quad \mathrm{Var}(\Psi_n) = \overset{\text{exercise}}{\cdots} \overset{(*)}{=} \frac{\sigma^2}{n}$

$*$
$$E[\Psi_n^2] = E[\Psi_n \cdot \Psi_n] = E\left[\left(\frac{1}{n}\sum_{j=1}^{n} X_j\right)\cdot\left(\frac{1}{n}\sum_{k=1}^{n} X_k\right)\right]$$
$$= E\left[\frac{1}{n^2}\sum_{j=1}^{n}\sum_{k=1}^{n} X_j X_k\right] = \frac{1}{n^2}\sum_{j=1}^{n}\sum_{k=1}^{n} E[X_j X_k]$$
$$= \frac{1}{n^2}\left[n(\sigma^2+\mu^2) + n(n-1)\mu^2\right]$$
$$= \cdots = \frac{\sigma^2}{n} + \mu^2$$
$$\therefore \mathrm{var}(\Psi_n) = \frac{\sigma^2}{n} + \mu^2 - (\mu)^2 = \frac{\sigma^2}{n}$$

Recall...

28.16

Applying the Chebyshev Inequality to Y_n, we get

$$P(\{|Y_n - \mu| \geq \varepsilon\}) \leq \frac{var(Y_n)}{\varepsilon^2} = \frac{\sigma^2}{n\varepsilon^2}.$$

$$\therefore \quad P(\{|Y_n - \mu| \geq \varepsilon\}) \leq \frac{\sigma^2}{n\varepsilon^2} \to 0 \text{ as } n \to \infty$$

$$\Rightarrow \quad Y_n \xrightarrow{(p)} \mu \qquad \text{as } n \to \infty \quad \blacksquare$$

28.17

<u>Corollary</u>: Suppose we repeat a simple experiment $(S_0, \mathcal{F}_0, P_0)$ many times independently.

Let $A \in \mathcal{F}_0$, and define a RV

$$X_k \triangleq \mathbb{1}_A(\omega) = \begin{cases} 1, & \omega \in A \\ 0, & \omega \notin A \end{cases}$$

on the k-th repetition of the experiment

<u>Note</u>: $X_1, \ldots, X_n, \ldots$ is an i.i.d. sequence of RVs with

$$E[X_k] = E[\mathbb{1}_A(\omega)] = P(A)$$

$$var(X_k) = \sigma^2 = P(A)(1 - P(A)).$$

28.18

Define

$$r_n(A) \triangleq \frac{X_1 + X_2 + \cdots + X_n}{n}$$

$$= \frac{1}{n} \sum_{k=1}^{n} X_k$$

$$E[r_n(A)] = E\left[\frac{X_1 + X_2 + \cdots X_n}{n}\right]$$

$$= \frac{1}{n} \underbrace{(P(A) + P(A) + \cdots P(A))}_{n\text{-times}} = \underline{P(A)}$$

Applying the WLLN to $r_n(A)$, we get

$$P(\{|r_n(A) - P(A)| \geq \varepsilon\}) \longrightarrow 0 \quad \text{as } n \to \infty$$

for all $\varepsilon > 0$ $\left(P(\{|r_n(A) - P(A)| \geq \varepsilon\}) \leq \frac{P(A)(1-P(A))}{n\varepsilon^2} \right.$

$\rightarrow 0$ as $n \to \infty$.

28.19

So this says that the relative frequency

$$r_n(A) = \frac{\left(\begin{array}{c}\text{number of times } A \text{ occurs} \\ \text{in } n \text{ independent trials}\end{array}\right)}{n}$$

converges in probability (p) to the probability $P(A)$.

$$r_n(A) \xrightarrow{(p)} P(A) \quad \text{as } n \to \infty.$$

Weak Law of Large Numbers

28.20

$$Y_n \xrightarrow{(p)} \mu \quad \text{as} \quad n \to \infty.$$

(You can show this even if the variance doesn't exist (i.e., is unbounded))
- proof is harder.

There are also stronger forms of the Law of Large numbers

Weak: convergence (p)

Strong: convergence (a.e.)

Strong Law of Large Numbers (Borel)

28.21

Let $\{X_n\}$ be a sequence of identically distributed RVs with mean μ and variance σ^2, and assume the RVs are uncorrelated:

$$E[(X_j-\mu)(X_k-\mu)]=0, \quad j \neq k$$

(So the X_j are not necessarily independent.)

Then

$$Y_n = \frac{1}{n}\sum_{k=1}^{n} X_k \xrightarrow{(a.e)} \mu \quad \text{as } n \to \infty$$

Proof: Beyond this course.

The Central Limit Theorem

28.22

13

Let $\{X_n\}$ be a sequence of i.i.d. RVs with mean μ and variance σ^2.

Define
$$Z_n \triangleq \frac{(X_1 + X_2 + \dots + X_n) - n\mu}{\sigma\sqrt{n}}, \quad n = 1, 2, \dots .$$

Then $\{Z_n\}$ converges in distribution to a RV Z that is Gaussian with mean 0 and variance 1.

i.e., $$F_{Z_n}(z) \longrightarrow \Phi(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-x^2/2}\,dx \quad \text{as } n \to \infty \quad \forall z \in \mathbb{R}.$$

28.23

14

Sketch of proof: We will show that

$$\underbrace{\Phi_{Z_n}(\omega)}_{\text{char. ftn. of } Z_n.} \xrightarrow{n \to \infty} \underbrace{e^{-\frac{1}{2}\omega^2}}_{\text{char. ftn. of } Z}, \quad \forall \omega \in \mathbb{R}$$

$$\begin{aligned}
\Phi_{Z_n}(\omega) &= E\left[e^{i\omega Z_n}\right] \\
&= E\left[\exp\left\{\frac{i\omega}{\sigma\sqrt{n}} \sum_{k=1}^{n} (X_k - \mu)\right\}\right] \\
&= E\left[\prod_{k=1}^{n} e^{i\omega(X_k - \mu)/\sigma\sqrt{n}}\right] \\
&= \prod_{k=1}^{n} E\left[e^{i\omega(X_k - \mu)/\sigma\sqrt{n}}\right] = \dots
\end{aligned}$$

$$= \left(E\left[\underbrace{e^{iw(X-\mu)/\sigma\sqrt{n}}}\right]\right)^n$$

We can expand this exponential as a power series (in w about $w=0$)

$\Downarrow$

$$E\left[e^{iw(X-\mu)/\sigma\sqrt{n}}\right]$$

$$= E\left[1 + \frac{iw(X-\mu)}{\sigma\sqrt{n}} + \frac{(iw)^2}{2n\sigma^2}(X-\mu)^2 + \underbrace{R(w)}_{\text{remainder}}\right]$$

$$= 1 + \frac{iw}{\sigma\sqrt{n}}\cancelto{0}{E[X-\mu]} + \frac{(iw)^2}{2n\sigma^2}\underbrace{E[(X-\mu)^2]}_{\sigma^2} + E[R(w)]$$

It can be shown that

$$\frac{E[R(w)]}{w^2/2n} \longrightarrow 0 \quad \text{as } n\to\infty,\ \forall w \in \mathbb{R}$$

(It's a bit of work!)

Thus we have

$$E\left[e^{iw(X-\mu)/\sigma\sqrt{n}}\right] = 1 - \frac{w^2}{2n\cancel{\sigma^2}}\cdot\cancel{\sigma^2} + E[R(w)]$$

$$\cong 1 - \frac{w^2}{2n}, \quad \text{as } n\to\infty.$$

$$\therefore\ \Phi_{Z_n}(w) \cong \left[1 - \frac{w^2}{2n}\right]^n \longrightarrow e^{-w^2/2} \quad \text{as } n\to\infty$$

(Recall: $\left(1+\frac{x}{n}\right)^n \to e^x$ as $n\to\infty$.)

$$\therefore\ \Phi_{Z_n}(w) \longrightarrow e^{-w^2/2}, \quad \forall w\in\mathbb{R} \qquad \left[1-\frac{w^2}{2n}\right]^n$$

$$\Rightarrow\ F_{Z_n}(z) \longrightarrow \Phi(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-x^2/2}\,dx$$

as $n\to\infty$.

28.26

General Forms of CLT:

We have assumed that $\{X_n\}$ are i.i.d. for our proof, but a CLT will hold even if the $\{X_n\}$ are not i.i.d. (i.e., they can be correlated and come from different distributions)

Very general form of CLT:

Lindeberg - Feller Central Limit Theorem

(See: P. Billingsley, *Probability and Measure* for conditions and proof. (Math / Stat 538)

28.27

ECE 600 Final Exam

Thursday, May 8, 2025

8:00 – 10:00 am

Room ARMS 1010 { For FNY and ONC (overflow) Sections

This will be a comprehensive exam.

This is a closed book, closed notes exam.

You may not use a calculator.

Start each problem on a new page. 28.28

- 10 "simple" multiple choice* questions **50%**
  - * There will be room to show your work.
- Two "work-out" problems **34%**
- One page of True/False Questions **16%**

EPE Online Section Answer Form

ECE600 Random Variables and Waveforms
Spring ~~2022~~ 2025

Mark R. Bell
MSEE 336

Final Exam Online Section Answer Form
~~May 4, 2022~~
May 8, 2025

1-10	
11	
12	
13	
Total	

$$R(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)e^{j\omega\tau}\,d\omega \leftrightarrow S(\omega) = \int_{-\infty}^{\infty} R(\tau)e^{-j\omega\tau}\,d\tau$$

$$\delta(\tau) \leftrightarrow 1 \qquad 1 \leftrightarrow 2\pi\delta(\omega)$$

$$e^{j\beta\tau} \leftrightarrow 2\pi\delta(\omega-\beta) \qquad \cos\beta\tau \leftrightarrow \pi\delta(\omega-\beta)+\pi\delta(\omega+\beta)$$

$$e^{-\alpha|\tau|} \leftrightarrow \frac{2\alpha}{\alpha^2+\omega^2} \qquad e^{-\alpha\tau^2} \leftrightarrow \sqrt{\frac{\pi}{\alpha}}\,e^{-\omega^2/4\alpha}$$

$$e^{-\alpha|\tau|}\cos\beta\tau \leftrightarrow \frac{\alpha}{\alpha^2+(\omega-\beta)^2}+\frac{\alpha}{\alpha^2+(\omega+\beta)^2}$$

$$2e^{-\alpha\tau^2}\cos\beta\tau \leftrightarrow \sqrt{\frac{\pi}{\alpha}}\left[e^{-(\omega-\beta)^2/4\alpha}+e^{-(\omega+\beta)^2/4\alpha}\right]$$

$$\begin{cases} 1-\frac{|\tau|}{T} & |\tau|<T \\ 0 & |\tau|>T \end{cases} \leftrightarrow \frac{4\sin^2(\omega T/2)}{T\omega^2}$$

$$\frac{\sin\sigma\tau}{\pi\tau} \leftrightarrow \begin{cases} 1 & |\omega|<\sigma \\ 0 & |\omega|>\sigma \end{cases}$$

Name: ______________________

ID #: ______________________

Directions:

1. **Print** your name and student number on the cover page.
2. Exam is closed book, closed notes, and no calculators.
3. Clearly designate all answers asked for (arrows, underline, box, etc.)

Problems 1–10 are multiple choice problems worth 5 points each. For each problem, write the lettercorresponding to the best answer next to the problem number. Space is provided to work out your solution for each of these problems. Please show your work! If your final grade is near a borderline, the quality of your written solutions could significantly impact your final grade.

1. Answer: $\boxed{C}$

$$\Phi_X(\omega) = E[e^{i\omega X}]$$

$$= \sum_{k=0}^{n} e^{i\omega k} \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \sum_{k=0}^{n} \binom{n}{k} (pe^{i\omega})^k (1-p)^{n-k}$$

Binomial Thm.

$$= (pe^{i\omega} + 1 - p)^n$$

$$= (1 + p(e^{i\omega} - 1))^n$$

etc.

2. Answer:

10. Answer:

11. Problems 11 is made up of 8 True/False questions, worth 2 points each. Fill in your answers T (true) or F (false) below, corresponding to the statements A–H in problem 11 on the exam.

A. ____

B. ____

C. ____

D. ____

E. ____

F. ____

G. ____

H. ____

Problems 12 and 13 are "work out" problems for which partial credit will be awarded for correctly reasoned work. It is important that you coherently present your thinking in the solution of these problems if you wish to receive partial credit (or full credit for that matter.) Please work problems 12 and 13 of the exam in the designated space below.

12. Problem 12 Solution:

(Problem 12 Solution Continued)

13. Problem 13 Solution:

(Problem 13 Solution Continued)

Stochastic Processes

- The idea of a stochastic process is a straightforward extension of the idea of a random variable.
- Instead mapping each outcome $\omega \in \mathcal{S}$ to a number $X(\omega)$, we map it to a function of time $X(t,\omega)$

Stochastic Processes

Defn(1): A random process defined on a random experiment $(\mathcal{S}, \mathcal{F}, P)$ is a family $\{X(t); t \in \mathbb{R}\}$ of random variables defined on $(\mathcal{S}, \mathcal{F}, P)$ and indexed by t.

Recall: A RV $X : \mathcal{S} \to \mathbb{R}$ for a particular outcome $\omega_0 \in \mathcal{S}$ is just a real number $X(\omega_0)$.

A random process $X(t, \omega)$ is a function of two variables: time and the outcome $\omega \in \mathcal{S}$.

Random process 28.31

$$\mathbf{X}(t, \omega) : \mathbb{R} \times \mathcal{S} \to \mathbb{R}$$

real time values — outcomes of a random experiment

Just as we often write the random variable $\mathbf{X}(\omega)$ as $\mathbf{X}$ (hiding explicit dependence on $\omega \in \mathcal{S}$)

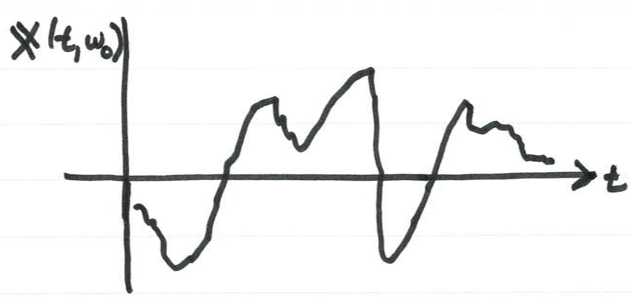
so we often write the random process $\mathbf{X}(t, \omega)$ as $\mathbf{X}(t)$. (again hiding the explicit dependence on $\omega \in \mathcal{S}$).

The boldface notation $\mathbf{X}$ or $\mathbf{X}(t)$ tells us there is an implicit dependence on $\omega \in \mathcal{S}$.

If we fix a particular $\omega_0 \in \mathcal{S}$ then we have 28.32

$$\mathbf{X}(\cdot, \omega_0) : \mathbb{R} \to \mathbb{R}$$

t



For each $\omega \in \mathcal{S}$, $\mathbf{X}(\cdot, \omega)$ is a different function of time.

28.23

Defn: For a fixed outcome $\omega_0 \in \mathcal{S}$, the time function $x(\cdot) = \mathbb{X}(\cdot, \omega_0)$ is called the <u>sample path</u> of the random process $\mathbb{X}(t)$ corresponding to ω_0. Each $\omega \in \mathcal{S}$ has a sample path $x(\cdot) = \mathbb{X}(\cdot, \omega)$. The set of all sample paths

$$\mathcal{E} = \{x(\cdot) : x(\cdot) = \mathbb{X}(\cdot, \omega), \text{ for } \omega \in \mathcal{S}\}$$

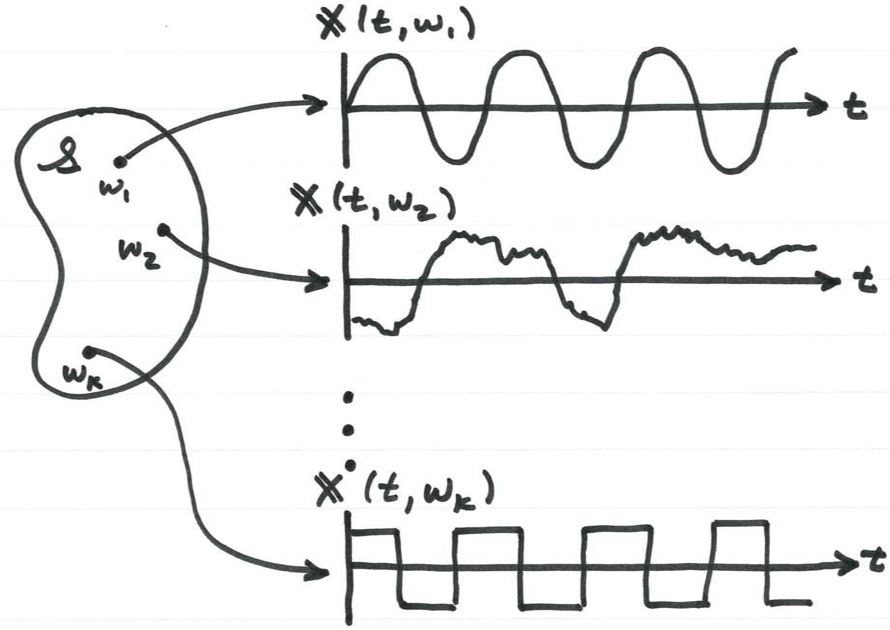
is called the <u>ensemble</u> of the R.P. $\mathbb{X}(t)$.

Having established these ideas, we can give the following equivalent definition of a random process ...

28.24

Defn$^{(2)}$: Consider a random experiment $(\mathcal{S}, \mathcal{F}, P)$. A function $\mathbb{X} : \mathcal{S} \to \mathcal{E}$, where $\mathcal{E}$ is a set of functions of time (sample paths) is called a <u>random process</u>. That is, $\mathbb{X}$ is a mapping assigning a function of time $\mathbb{X}(\cdot, \omega)$ to each $\omega \in \mathcal{S}$.

A picture appears as follows:



<u>Example</u>: Roll a die:

$$\mathcal{S} = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6\}$$

1 2 3 4 5 6

We know how to construct $(\mathcal{S}, \mathcal{F}, P)$.

Now construct a random process $X(t)$ as

$$X(t, \omega_k) = \cos(2\pi k t), \quad k = 1, 2, \ldots, 6.$$

numerical outcome on die

So we now have a random process $X(t)$ that is a cosine function whose frequency is selected at random:

(example continued)

$$\mathbb{X}(t, \omega_k) = \cos(2\pi k t)$$

ω	$\mathbb{X}(t,\omega)$
ω_1	$\cos 2\pi t$
ω_2	$\cos 4\pi t$
ω_3	$\cos 6\pi t$
ω_4	$\cos 8\pi t$
$\vdots$	
ω_6	$\cos 12\pi t$