Session 14

14.1

Matched Filters Mismatched in Delay and Doppler: The Ambiguity Function

ECE678: Radar Engineering
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Matched-Filter Mismatched Response

Transmit: $s(t)$

Receive: $r(t) = s(t-\tau_0)e^{i2\pi\nu_0 t}$

$\tau_0 =$ actual delay in received signal.

$\nu_0 =$ actual Doppler shift in received signal.

Assume we process the received signal with a matched filter $h_{\tau,\nu}(t)$ matched to the signal

$$s_{\tau,\nu}(t) = s(t-\tau)e^{i2\pi\nu t}.$$

Then the impulse response of the matched filter designed to be sampled at time $t = T+\tau$ is

$$\begin{aligned} h_{\tau,\nu}(t) &= s^*_{\tau,\nu}(T+\tau-t) \\ &= s^*(T+\tau-t-\tau)e^{-i2\pi(T+\tau-t)}. \end{aligned}$$

The output of the matched filter at time $t = T+\tau$ is

$$O_{T+\tau}(\tau,\nu) = r(t) * h_{\tau,\nu}(t)\Big|_{t=T+\tau} = s_{\tau_0\nu_0}(t) * h_{\tau,\nu}(t)\Big|_{t=T+\tau}$$

$$= \int_{-\infty}^{\infty} s(p-\tau_0)e^{i2\pi\nu_0 t} s^*(T+\tau-(t-p)-\tau) \cdot e^{-i2\pi\nu(T+\tau-(t-p))}dp \Big|_{t=T+\tau}$$

$$= \int_{-\infty}^{\infty} s(p-\tau_0)s^*(p-\tau)e^{i2\pi\nu_0 p}e^{-i2\pi\nu p}\,dp$$

let $x = p-\tau_0 \Rightarrow p = x+\tau_0 \Rightarrow dp = dx$

$$= \int_{-\infty}^{\infty} s(x)s^*(x-(\tau-\tau_0))e^{-i2\pi(\nu-\nu_0)(x+\tau_0)}\,dx$$

$$= e^{-i2\pi(\nu-\nu_0)\tau_0}\int_{-\infty}^{\infty} s(x)s^*(x-(\tau-\tau_0))e^{-i2\pi(\nu-\nu_0)x}\,dx$$

$$= e^{-i2\pi(\nu-\nu_0)\tau_0}\cdot\beta_s(\tau-\tau_0,\nu-\nu_0)$$

where

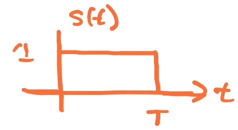
$$\beta_s(\tau,\nu) \triangleq \int_{-\infty}^{\infty} s(t)s^*(t-\tau)e^{-i2\pi\nu t}\,dt,$$

which is an ambiguity function of $s(t)$

Example: Ambiguity Function of a Rectangular Pulse

Let

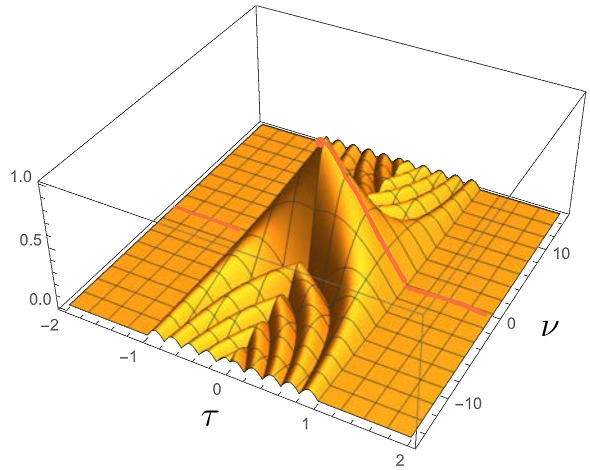
$$s(t) = 1_{[0,T]}(t).$$

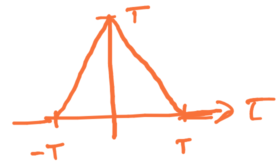


Then it can be shown (exercise) that

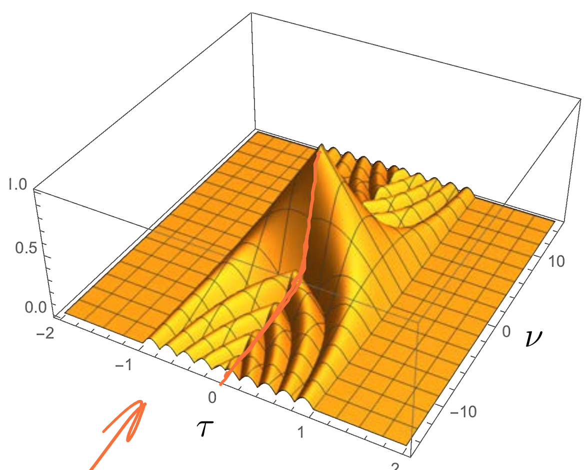
$$\beta_s(\tau,\nu) = e^{-i\pi(T+\tau)}(T-|\tau|)\frac{\sin \pi\nu(T-|\tau|)}{\pi\nu(T-|\tau|)} \cdot 1_{[-T,T]}(\tau).$$

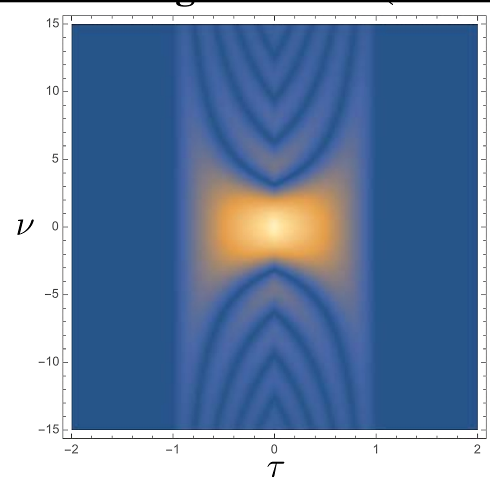
A plot of $|\beta_s(\tau,\nu)|$ for a pulse of duration $T = 1$ appear as follows:

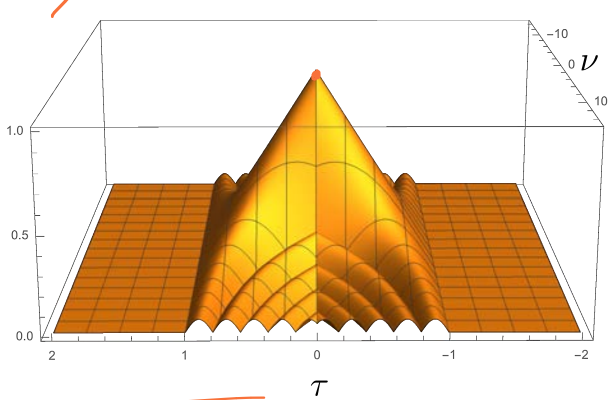


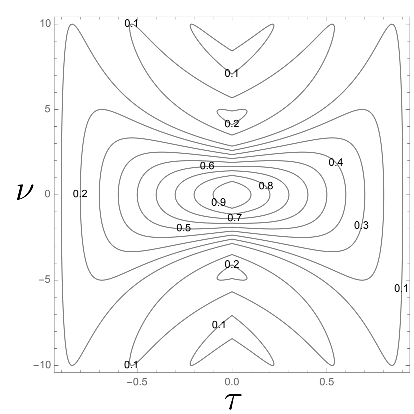


Example: Ambiguity Function of a Rectangular Pulse (Cont.)









Ambiguity Function Definitions

Definition:(Asymmetric ambiguity function) The *asymmetric ambiguity function* of a finite energy signal $s(t)$ is defined as

$$\beta_s(\tau,\nu) = \int_{-\infty}^{\infty} s(t)s^*(t-\tau)e^{-i2\pi\nu t}\,dt.$$

Definition:(Symmetric ambiguity function) The *symmetric ambiguity function* of a finite energy signal $s(t)$ is defined as

$$\Gamma_s(\tau,\nu) = \int_{-\infty}^{\infty} s(t+\tau/2)s^*(t-\tau/2)e^{-i2\pi\nu t}\,dt.$$

- The adjectives *asymmetric* and *symmetric* follow from the way in which the delay τ is distributed in the integrand of the respective definitions.
- The asymmetric ambiguity function is the form most often used by radar engineers, primarily because it arises in determining the response of a matched filter radar as we have seen.
- The symmetric ambiguity function is more often used in theoretical investigations of signal properties because its symmetric form simplifies some derivations, as well as the fact that it is closely related to the widely used Wigner distribution of time-frequency analysis.

Ambiguity Functions (Cont.)

It can easily be shown that the symmetric and asymmetric ambiguity functions are related by the expressions

$$\Gamma_s(\tau,\nu) = e^{i\pi\nu\tau}\cdot\beta_s(\tau,\nu)$$

and

$$\beta_s(\tau,\nu) = e^{-i\pi\nu\tau}\cdot\Gamma_s(\tau,\nu).$$

It can also be shown that the symmetric ambiguity function can be written in terms of the Fourier transform $S(f)$ of $s(t)$ as

$$\Gamma_s(\tau,\nu) = \int_{-\infty}^{\infty} S(f+\nu/2)S^*(f-\nu/2)e^{i2\pi\tau f}\,df,$$

and the asymmetric ambiguity function can be written as

$$\beta_s(\tau,\nu) = \int_{-\infty}^{\infty} S(f+\nu)S^*(f)e^{i2\pi\tau f}\,df.$$

Ambiguity Functions (Cont.)

Definition: The modulus (magnitude) of either of the above ambiguity functions is called the *ambiguity surface*.

Note: Since

$$\Gamma_s(\tau, \nu) = e^{i\pi\nu t} \cdot \beta_s(\tau, \nu),$$

it follows that

$$|\Gamma_s(\tau, \nu)| = |\beta_s(\tau, \nu)|.$$

We will write the ambiguity surface as $\mathcal{A}_s(\tau, \nu) = |\beta_s(\tau, \nu)| = |\Gamma_s(\tau, \nu)|$.

Because the ambiguity function is a complex valued function, it is difficult to plot and visualize.

The *ambiguity surface*, being real valued, is easy to plot and visualize.

The *ambiguity surface* is also meaningful, as it tells us the *amplitude* of the mismatched matched-filter response.

Why two forms?

- Mathematically, symmetric form can be less cumbersome, especially for symmetric signals.
- The symmetric ambiguity functions is the *two dimensional Fourier Transform* of the *Wigner Distribution.*
- Historically, an asymmetric form was first introduced by Woodward.
- Numerically, the asymmetric form is easier to compute for a sampled signal (*i.e.,* no "half-sample offsets".)
- In practice, use the most convenient form for your computation, because moving between the two forms is easy:

$$\Gamma_s(\tau, \nu) = e^{i\pi\nu\tau} \cdot \beta_s(\tau, \nu)$$

$$\beta_s(\tau, \nu) = e^{-i\pi\nu\tau} \cdot \Gamma_s(\tau, \nu).$$

Note that:

1. $$\beta_s(\tau,0) = \Gamma_s(\tau,0) = \int_{-\infty}^{\infty} s(t)s^*(t-\tau)\,dt = \int_{-\infty}^{\infty} s(\gamma+\tau)s^*(\gamma)\,d\gamma = \text{“time autocorrelation function of } s(t)\text{.”}$$

2. $$\beta_s(0,\nu) = \Gamma_s(0,\nu) = \int_{-\infty}^{\infty} |s(t)|^2 e^{-i2\pi\nu t}\,dt = \text{“Fourier transform of } |s(t)|^2\text{.”}$$

3. $$\beta_s(0,0) = \Gamma_s(0,0) = \int_{-\infty}^{\infty} |s(t)|^2\,dt = \text{“Energy in signal } s(t)\text{.”}$$

Theorem: The Symmetric ambiguity function $\Gamma_s(\tau,\nu)$ of $(s(t)$ can be written as

$$\Gamma_s(\tau,\nu) = \int_{-\infty}^{\infty} S(f+\nu/2)S^*(f-\nu/2)e^{i2\pi f\tau}\,df,$$

where

$$S(f) = \int_{-\infty}^{\infty} s(t)e^{-i2\pi f\tau}\,dt.$$

Proof: (exercise)

Corollary:

$$\int_{-\infty}^{\infty} \Gamma_s(\tau,\nu)e^{i2\pi\nu t}\,d\nu = s(t+\tau/2)s^*(t-\tau/2),$$

$$\int_{-\infty}^{\infty} \Gamma_s(\tau,\nu)e^{-i2\pi\nu t}\,d\tau = S(f+\nu/2)S^*(f-\nu/2).$$

14.12

Corollary:

$$\int_{-\infty}^{\infty} \Gamma_s(\tau,\nu) e^{i2\pi\nu t}\, d\nu = s(t+\tau/2)s^*(t-\tau/2),$$

$$\int_{-\infty}^{\infty} \Gamma_s(\tau,\nu) e^{-i2\pi\nu t}\, d\tau = S(f+\nu/2)S^*(f-\nu/2).$$

From the first line of the Corollary:

$$\int_{-\infty}^{\infty} \Gamma_s(\tau,\nu) e^{i2\pi\nu t}\, d\nu = s(t+\tau/2)s^*(t-\tau/2),$$

Substituting $t \mapsto \tau/2$ yields

$$s(\tau) = \frac{1}{s^*(0)} \int_{-\infty}^{\infty} \Gamma_s(\tau,\nu) e^{i\pi\nu t}\, d\nu$$

and

$$|s(0)|^2 = \int_{-\infty}^{\infty} \Gamma_s(0,\nu)\, d\nu.$$

$\Rightarrow$ $s(t)$ can be recovered from $\Gamma_s(\tau,\nu)$.

14.13

Note: The inverse two-dimensional Fourier transform of $\Gamma(\tau,\nu)$ can be written as

$$\mathcal{W}(t,f) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Gamma(\tau,\nu)\, e^{i2\pi\nu t}\, e^{i2\pi f\tau}\, d\nu\, d\tau$$

$$= \int_{-\infty}^{\infty} s(t+\tau/2)s^*(t-\tau/2)\, e^{i2\pi f\tau}\, d\tau$$

This is the Wigner Distribution of $s(t)$—a popular *time-frequency distribution* in signal theory.

$\mathcal{W}(t,f)$ superficially resembles $\Gamma(\tau,\nu)$, but they are actually quite different.

Properties of the Ambiguity Function

Ambiguity functions have many interesting properties.

Among all functions $f : \mathbf{R}^2 \to \mathbf{C}$,
ambiguity functions are quite rare.

In radar waveform design problems, we often know the characteristics of the ambiguity function we would like.

We then want to design a waveform having an ambiguity function that approximates it.

The Synthesis Problem—very difficult!

Properties of the Ambiguity Function

- Because synthesis is hard, another approach is to study the ambiguity functions of many different classes of functions.
- This helps you develop intuition about how different forms of modulation effect the ambiguity function.
- This intuition is greatly aided by an understanding of the properties of ambiguity functions.

Properties of the Ambiguity Function

Property 1 *(Time-shift property) Let* $v(t) = s(t - \Delta)$*, where* $\Delta \in \mathbf{R}$*. Then*

$$\beta_v(\tau, \nu) = e^{-i2\pi\nu\Delta}\beta_s(\tau, \nu).$$

Proof: The ambiguity function of $v(t) = s(t - \Delta)$ can be written as

$$\begin{aligned}
\beta_v(\tau, \nu) &= \int_{-\infty}^{\infty} v(t)v^*(t-\tau)e^{-i2\pi\nu t}\,dt \\
&= \int_{-\infty}^{\infty} s(t-\Delta)s^*(t-\Delta-\tau)e^{-i2\pi\nu t}\,dt \\
&= \int_{-\infty}^{\infty} s(x)s^*(x-\tau)e^{-i2\pi\nu(x+\Delta)}\,dx, \qquad \text{letting } x = t - \Delta, \\
&= e^{-i2\pi\nu\Delta}\int_{-\infty}^{\infty} s(x)s^*(x-\tau)e^{-i2\pi\nu x}\,dx \\
&= e^{-i2\pi\nu\Delta}\beta_s(\tau, \nu).
\end{aligned}$$

Properties of the Ambiguity Function (Cont.)

Property 2 *(Frequency-shift property)* Let $v(t) = s(t)e^{i2\pi ft}$, where $f \in \mathbf{R}$. Then

$$\beta_v(\tau, \nu) = e^{-i2\pi f\tau}\beta_s(\tau, \nu).$$

Proof: The ambiguity function of $v(t) = s(t)e^{i2\pi ft}$ can be written as

$$\begin{aligned}
\beta_v(\tau, \nu) &= \int_{-\infty}^{\infty} v(t)v^*(t-\tau)e^{-i2\pi\nu t}\,dt \\
&= \int_{-\infty}^{\infty} s(t)e^{i2\pi ft}(s(t-\tau)e^{i2\pi f(t-\tau)})^*e^{-i2\pi\nu t}\,dt \\
&= e^{i2\pi f\tau}\int_{-\infty}^{\infty} s(t)s^*(t-\tau)e^{-i2\pi\nu t}\,dt, \\
&= e^{i2\pi f\tau}\beta_s(\tau, \nu).
\end{aligned}$$

Properties of the Ambiguity Function (Cont.)

14.18

Property 3 *(Symmetry property)*

$$\beta_s(-\tau,-\nu) = e^{-i2\pi\nu\tau}\beta_s^*(\tau,\nu).$$

Proof: We can evaluate $\beta_s(-\tau,-\nu)$ as

$$\begin{aligned}
\beta_s(-\tau,-\nu) &= \int_{-\infty}^{\infty} s(t)s^*(t-(-\tau))e^{-i2\pi(-\nu)t}\,dt \\
&= \int_{-\infty}^{\infty} s(t)s^*(t+\tau)e^{+i2\pi\nu t}\,dt \\
&= \left(\int_{-\infty}^{\infty} s(t+\tau)s^*(t)e^{-i2\pi\nu t}\,dt\right)^* \\
&= \left(\int_{-\infty}^{\infty} s(x)s^*(x-\tau)e^{-i2\pi\nu(x-\tau)}\,dx\right)^*, \quad (\text{letting } x = t+\tau) \\
&= e^{-i2\pi\nu\tau}\left(\int_{-\infty}^{\infty} s(x)s^*(x-\tau)e^{-i2\pi\nu x}\,dx\right)^* \\
&= e^{-i2\pi\nu\tau}\beta_s^*(\tau,\nu).
\end{aligned}$$

14.19

Property 4 *(Maximum property)* The largest value of $|\beta_s(\tau,\nu)|$ always occurs at the origin $(\tau,\nu) = (0,0)$:

$$|\beta_s(\tau,\nu)| \le \beta_s(0,0) = E_s,$$

where E_s is the energy in the signal $s(t)$.

Proof: Recall that the *Cauchy-Schwarz Inequality* states that for any two square-integrable functions $a(t)$ and $b(t)$,

$$\left|\int_{-\infty}^{\infty} a(t)b(t)\,dt\right|^2 \le \int_{-\infty}^{\infty} |a(t)|^2\,dt \cdot \int_{-\infty}^{\infty} |b(t)|^2\,dt,$$

with equality if and only if $b(t) = ka^*(t)$ (a.e.), where k is a constant. It follows that

$$\begin{aligned}
|\beta_s(\tau,\nu)|^2 &= \left|\int_{-\infty}^{\infty} \left[s(t)e^{-i\pi\nu t}\right]\left[s(t-\tau)e^{i\pi\nu t}\right]^*\,dt\right|^2 \\
&\le \int_{-\infty}^{\infty} \left|s(t)e^{-i\pi\nu t}\right|^2\,dt \cdot \int_{-\infty}^{\infty} \left|s(t-\tau)e^{i\pi\nu t}\right|^2\,dt \\
&= \int_{-\infty}^{\infty} |s(t)|^2\,dt \cdot \int_{-\infty}^{\infty} |s(t-\tau)|^2\,dt \\
&= E_s \cdot E_s \\
&= E_s^2,
\end{aligned}$$

where equality holds if and only if

$$\left[s(t-\tau)e^{i\pi\nu t}\right]^* = k\left[s(t)e^{-i\pi\nu t}\right]^*.$$

This can easily be seen to occur when $(\tau,\nu) = (0,0)$. Thus it follows that

$$|\beta_s(\tau,\nu)| \le \beta_s(0,0) = E_s.$$

Properties of the Ambiguity Function (Cont.)

Property 5 *(Time-scaling property)* Let $v(t) = s(\alpha t)$, where $\alpha \in \mathbf{R}$. Then

$$\beta_v(\tau, \nu) = \frac{1}{|\alpha|}\beta_s(\alpha\tau, \nu/\alpha).$$

Proof: The proof is left as an exercise.

Properties of the Ambiguity Function (Cont.)

Property 6 *(Quadratic phase shift (chirp) property)* Let

$$v(t) = s(t)e^{i\pi\alpha t^2},$$

where $\alpha \in \mathbf{R}$. Then

$$\beta_v(\tau, \nu) = e^{-i\pi\alpha\tau^2}\beta_s(\tau, \nu - \alpha\tau).$$

Proof: We note that $\beta_s(\tau, \nu)$ can be written as

$$\begin{aligned}
\beta_v(\tau, \nu) &= \int_{-\infty}^{\infty} v(t)v^*(t-\tau)e^{-i2\pi\nu t}\,dt \\
&= \int_{-\infty}^{\infty} s(t)e^{i\pi\alpha t^2}[s(t-\tau)e^{i\pi\alpha(t-\tau)^2}]^* \cdot e^{-i2\pi\nu t}\,dt \\
&= \int_{-\infty}^{\infty} s(t)e^{i\pi\alpha t^2}s^*(t-\tau)e^{-i\pi\alpha(t^2-2t\tau+\tau^2)}e^{-i2\pi\nu t}\,dt, \\
&= e^{-i\pi\alpha\tau^2}\int_{-\infty}^{\infty} s(t)s^*(t-\tau)e^{-i2\pi(\nu-\alpha\tau)t}\,dt, \\
&= e^{-i\pi\alpha\tau^2}\beta_s(\tau, \nu).
\end{aligned}$$