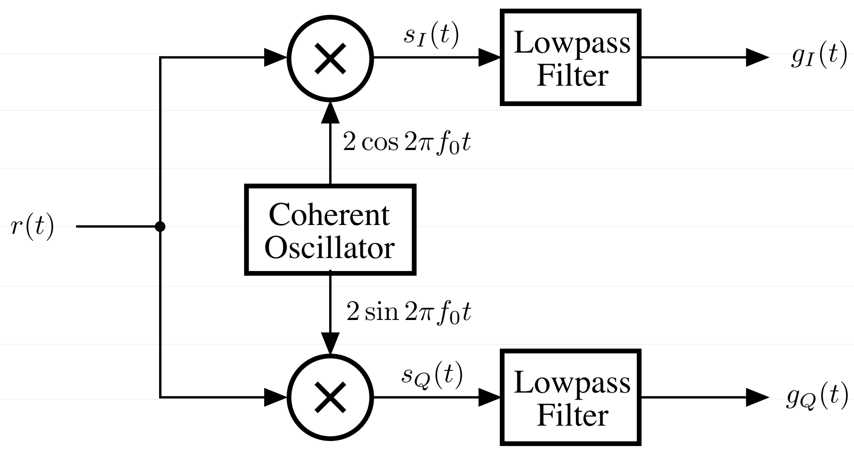
Session 7

Radar Signal Models

- Radar, like most radio systems, modulates a signal of interest onto a high-frequency RF carrier.
- The information bearing signal or pulse typically carries the information of interest.
- In a radar receiver, the received waveform's amplitude, phase and delay must be extracted from the received signal.

7.2

- One very common way of doing this is by "beating the signal down to baseband" using a quadrature detector:



7.3

Suppose a pulse $p(t) = 1_{[0,T]}(t)$ amplitude modulates a carrier with frequency f_0 to produce a signal $s(t)$:

$$s(t) = \cos(2\pi f_0 t) \cdot 1_{[0,T]}(t)$$

Now if the received signal $r(t)$ is a τ-delayed version of $s(t)$, we have

$$r(t) = s(t-\tau) = \cos[2\pi f_0 (t-\tau)] \cdot 1_{[0,T]}(t-\tau)$$
$$= \cos(2\pi f_0 t - 2\pi f_0 \tau) \cdot 1_{[\tau, T+\tau]}(t)$$

7.4

Basic Trig. Identities

$$\cos A \cdot \cos B = \frac{1}{2}\left(\cos(A+B) + \cos(A-B)\right)$$

$$\cos A \cdot \sin B = \frac{1}{2}\left(\sin(A+B) - \sin(A-B)\right)$$

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

Paul Nahin, Dr. Eulers Faboulous Formula

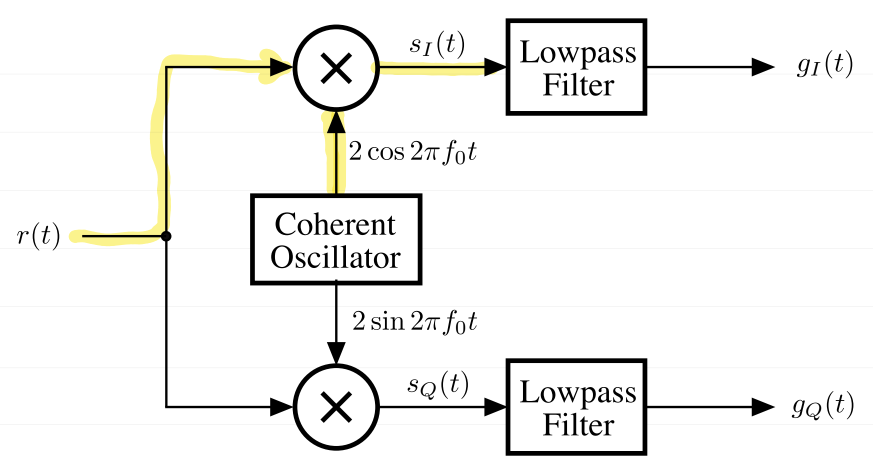
$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

e.g.

$$\cos A \cdot \cos B = \left(\frac{e^{iA} + e^{-iA}}{2}\right)\left(\frac{e^{iB} + e^{-iB}}{2}\right) = \cdots \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$$

7.5



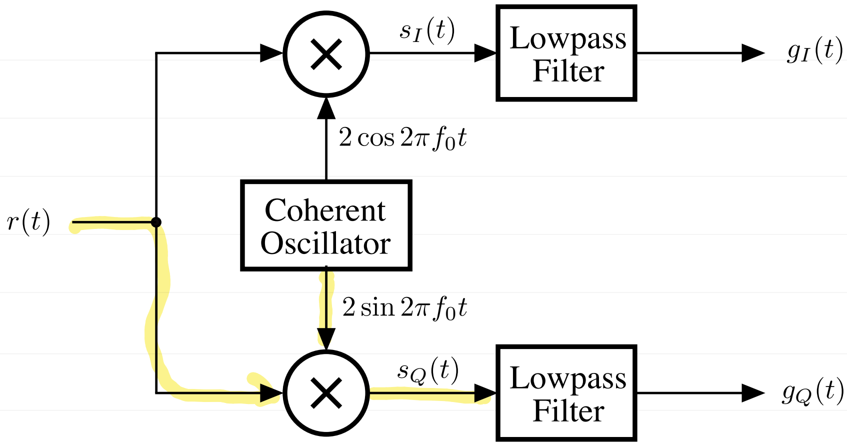
$$s_I(t) = r(t) \cdot 2\cos 2\pi f_0 t$$

$$= \underbrace{\cos(2\pi f_0 t - 2\pi f_0 \tau)}_{A} \cdot 1_{[\tau, T+\tau]} \cdot 2\underbrace{\cos 2\pi f_0 t}_{B}$$

$$= \left[\cos(2\pi(2f_0)t - 2\pi f_0\tau) + \cos(-2\pi f_0 \tau)\right] \cdot 1_{[\tau, T+\tau]}(t)$$

$$\left[\cos(2\pi(2f_0)t - 2\pi f_0\tau) + \cos(2\pi f_0 \tau)\right] \cdot 1_{[\tau, T+\tau]}(t)$$

7.6



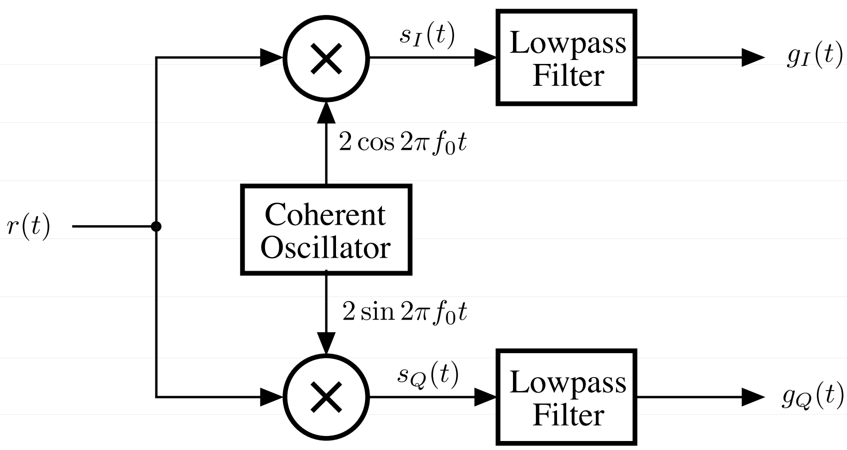
$$s_Q(t) = r(t)\cdot 2\sin(2\pi f_0 t)$$

$$= \cos(2\pi f_0 t - 2\pi f_0 \tau)\cdot 1_{[\tau,T+\tau]}(t) \cdot \sin(2\pi f_0 t)$$

$$= \left[\sin(2\pi(2f_0)t - 2\pi f_0\tau) - \sin(-2\pi f_0 \tau)\right]\cdot 1_{[\tau,T+\tau]}(t)$$

$$= \left[\sin(2\pi(2f_0)t - 2\pi f_0\tau) + \sin(2\pi f_0\tau)\right]\cdot 1_{[\tau,T+\tau]}(t)$$

7.7



$$g_I(t) = LPF\{s_I(t)\} = \cos(2\pi f_0\tau)\cdot 1_{[\tau,T+\tau]}(t)$$

"In phase" ~ real

$$g_Q(t) = LPF\{s_Q(t)\} = \sin(2\pi f_0 \tau)\cdot 1_{[\tau,T+\tau]}(t)$$

"Quadrature" ~ imaginary

7.8

We can construct a complex waveform

$$z(t) = g_I(t) + i\, g_Q(t)$$

$$= \cos(2\pi f_0 \tau)\cdot 1_{[\tau+T+\tau]}(t) + i \sin(2\pi f_0 \tau)\cdot 1_{[\tau,T+\tau]}(t)$$

$$= [\cos(2\pi f_0\tau) + i \sin(2\pi f_0 \tau)] \cdot 1_{[\tau,T+\tau]}(t)$$

$$= e^{i 2\pi f_0 \tau} \cdot 1_{[\tau,T+\tau]}(t)$$

This is a rectangular pulse of duration T starting at time τ.

It has phase $2\pi f_0 \tau$ radians.

7.9

The form

$$z(t) = e^{i 2\pi f_0 \tau} \cdot 1_{[\tau,T+\tau]}(t)$$

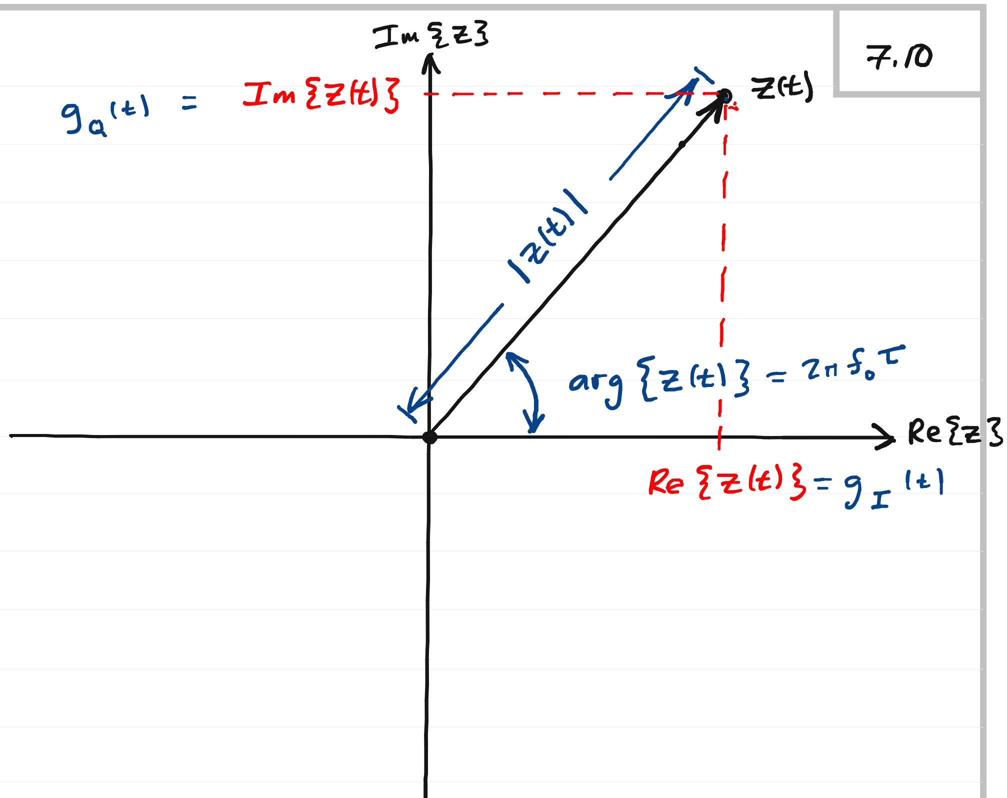
is called the complex baseband form of real received passband signal

$$r(t) = \cos(2\pi f_0 (t-\tau)) \cdot 1_{[0,T]}(t-\tau)$$

$$= \cos(2\pi f_0 t - 2\pi f_0 \tau) \cdot 1_{[\tau,T+\tau]}$$

We can represent $z(t)$ at time t using a phasor diagram:

7.10



The Narrowband Complex Baseband Signal

7.11

We now generalize the form of the received signal to be

$$r(t) = a(t) \cos(2\pi f_0 t + \theta(t))$$

where

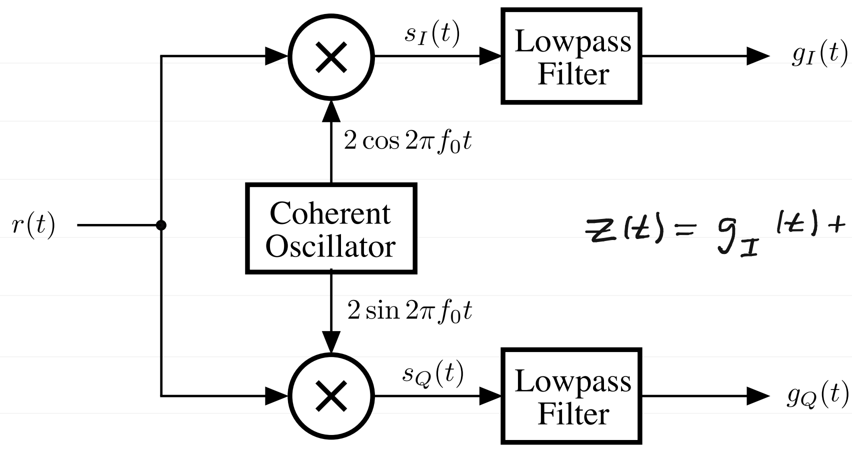
$a(t)$ = real-valued amplitude function

and

$\theta(t)$ = real-valued phase function.

Assume $a(t)$ and $\theta(t)$ vary slowly compared to $\cos 2\pi f_0 t$.

7.12



$$z(t) = g_I(t) + i\, g_Q(t)$$

$$g_I(t) = LPF\{s_I(t)\} = a(t)\cos\theta(t)$$

$$g_Q(t) = LPF\{s_Q(t)\} = -a(t)\sin\theta(t)$$

and we have the complex baseband waveform

$$z(t) = g_I(t) + i\, g_Q(t) = a(t)\, e^{-i\theta(t)}.$$

7.13

$$r(t) = a(t)\cos(2\pi f_0 t + \theta(t))$$

(real passband form)

$$\Updownarrow$$

$$z(t) = a(t)\, e^{-i\theta(t)}$$

(complex baseband form)

<u>n.b.</u> $r(t) = \mathrm{Re}\{ z(t)\, e^{i 2\pi f_0 t}\}.$

Narrowband Complex Baseband Noise

7.14

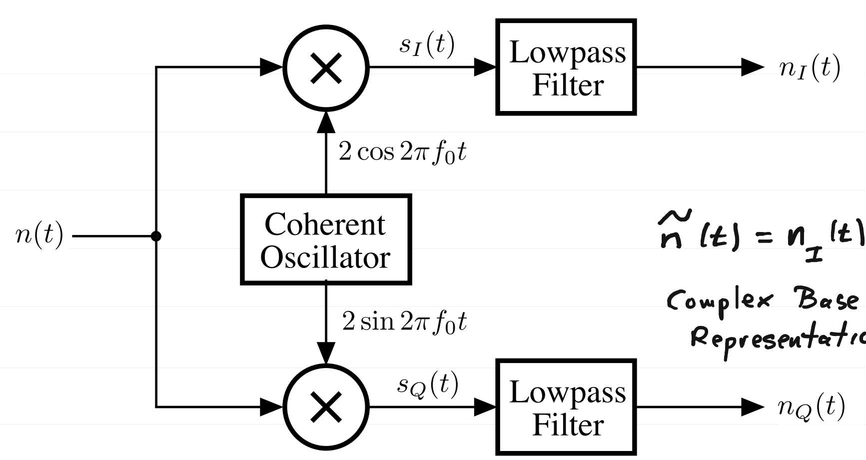
Consider a real, wide-sense stationary random process $n(t)$.

Assume that $n(t)$ is wide-sense stationary white Gaussian noise.

If we beat $n(t)$ down to baseband, we get a corresponding complex baseband waveform

$$\tilde{n}(t) = n_I(t) + i\, n_Q(t).$$

7.15

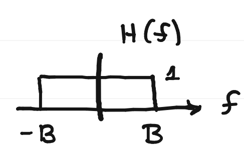


$$\tilde{n}(t) = n_I(t) + i\, n_Q(t)$$

Complex Baseband Representation

If we have ideal low-pass filters

$$H(f) = 1_{[-B,B]}(f),$$



then

$$S_{n_I n_I}(f) = S_{n_Q n_Q}(f) = \frac{N_0}{2}\, 1_{[-B,B]}(f).$$

7.16

Furthermore, $n_I(t)$ and $n_Q(t)$ have identical autocorrelation functions:

$$R_{n_I}(\tau) = R_{n_Q}(\tau) = \int \frac{N_0}{2} e^{i2\pi f\tau} df$$

$$= N_0 B \left(\frac{\sin 2\pi B\tau}{2\pi B\tau} \right)$$

$$= N_0 B \operatorname{sinc}(2B\tau)$$

where

$$\operatorname{sinc}(x) \triangleq \frac{\sin \pi x}{\pi x}.$$

7.17

Furthermore, it can be shown that

$$E[n_I(t+\tau) n_Q(t)] = -E[n_Q(t+\tau) n_I(t)]$$

$$\Rightarrow \quad R_{n_I n_Q}(\tau) = -R_{n_Q n_I}(\tau)$$

It follows that

$$R_{\tilde{n}}(\tau) = E[\tilde{n}(t+\tau)\tilde{n}^*(\tau)]$$

$$= E[(n_I(t+\tau) + i n_Q(t+\tau))(n_I(t) - i n_Q(t))]$$

$$= R_{n_I}(\tau) + R_{n_Q}(\tau) + i\left(E[n_Q(t+\tau)n_I(t)] - E[n_I(t+\tau)n_Q(t)]\right)$$

$$= 2\left(R_{n_I}(\tau) + R_{n_Q n_I}(\tau)\right).$$

From this, we make the following observations...

7.18

1. Since $R_{\tilde{n}}(0) = E[|\tilde{n}(0)|^2]$ is real,

$$E[n_I(t)\, n_Q(t)] = 0$$

$\Rightarrow$ $n_I(t) \perp n_Q(t)$ for all t (uncorrelated)

$\Rightarrow$ $n_I(t) \perp\!\!\!\perp n_Q(t)$, $\forall t$ if jointly Gaussian (independent)

2. From the above definitions

$$E[|\tilde{n}(t)|^2] = 2E[n^2(t)] = 2E[n_I^2(t)] = 2E[n_Q^2(t)].$$

$$\Rightarrow \quad E[n^2(t)] = E[n_I^2(t)] = E[n_Q^2(t)] = \frac{1}{2}E[|\tilde{n}(t)|^2]$$

7.19

3. If the power spectral density (PSD) of $n(t)$ is symmetric about the carrier frequency f_0, (as it is in this case), then the PSD of $\tilde{n}(t)$ is an even function

$\Rightarrow$ $R_{\tilde{n}}(\tau)$ is real for all τ.

$\Rightarrow$ $R_{n_Q n_I}(\tau) = 0$, $\forall \tau$

$\Rightarrow$ $n_I(t) \perp n_Q(t+\tau)$, $\forall t$, $\forall \tau$

$\Rightarrow$ $n_I(t_1) \perp n_Q(t_2)$, $\forall t_1$, $\forall t_2$

$\Rightarrow$ $n_I(t_1) \perp\!\!\!\perp n_Q(t_2)$, $\forall t_1$, $\forall t_2$ when they are jointly Gaussian

$\Rightarrow$ $n_I(t)$ and $n_Q(t)$ are independent Gaussian random processes in our case.

7.20

So going back to our pulse

$$r(t) = \cos(2\pi f_0 (t-\tau)) \cdot 1_{[\tau, T+\tau]}(t) + n(t)$$

$$z(t) = e^{i 2\pi f_0 \tau} \cdot 1_{[\tau, T+\tau]}(t) + \tilde{n}(t).$$

$$= (\cos 2\pi f_0 \tau, \sin 2\pi f_0 \tau) \cdot 1_{[\tau, T+\tau]}(t) + (n_I(t), n_Q(t))$$

$$= \underbrace{\left[\cos(2\pi f_0 \tau) \cdot 1_{(\tau, T+\tau)}(t) + n_I(t)\right]}_{\text{"In-phase" component}}$$

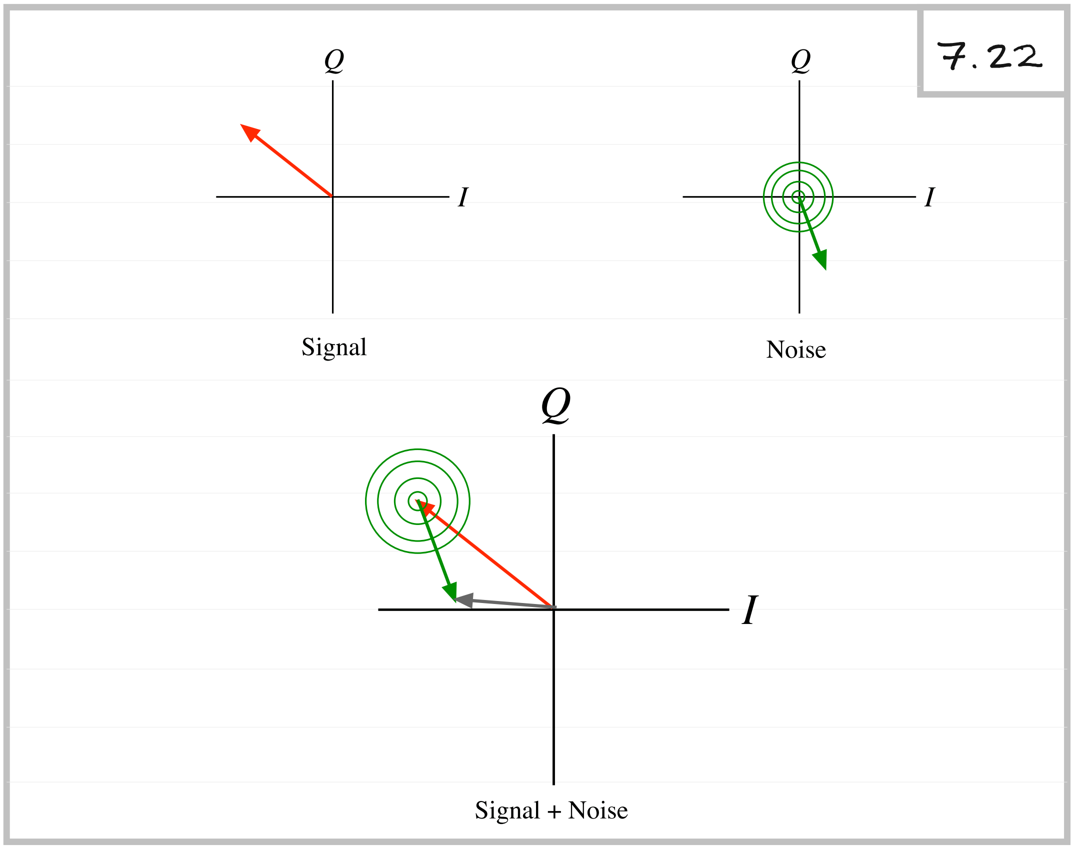
$$+ i \underbrace{\left[\sin(2\pi f_0 \tau) \cdot 1_{(\tau, T+\tau)}(t) + n_Q(t)\right]}_{\text{"Quadrature" component}}$$

7.21

For any t, $\tilde{n}(t) = n_I(t) + i n_Q(t)$ is a circular Gaussian process:

$$E[n_I(t)] = E[n_Q(t)] = 0$$

$$\mathrm{var}[n_I(t)] = \mathrm{var}[n_Q(t)] = \sigma^2 = N_0 B.$$


7.22
Q
I
Signal
Q
I
Noise
Q
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Signal + Noise