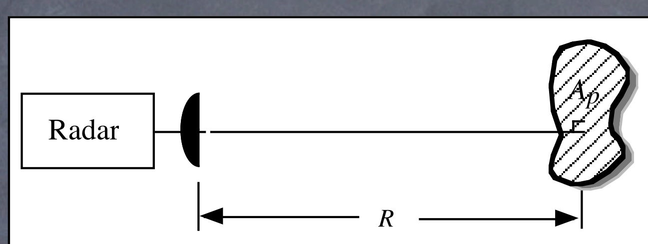
Session 5

Recall...

RCS of a Perfectly Conducting Plate

5.1

Consider a perfectly conducting plate with dimensions. much greater than λ. Assume area A_p.



Assume perpendicular orientation to incident wave.

In the far field, the plate is uniformly illuminated.

The plate reflects or radiates the wave as if it were a uniformly illuminated aperture of area A_p. (It is!)

5.2

$$\boxed{\sigma = \frac{4\pi A_p^2}{\lambda^2}.}$$

Applies only to a plate perpendicular to the direction of propagation.

If the plate is tilted, it will not be uniformly illuminated. There will be a phase shift across it.

This nonuniformly illuminator then transmits the wave back to the radar, which is off axis.

5.3

Using this approach, we can find that

For a square plate with dimensions $W \times W$ tilted by θ along an axis parallel to a side, we have

$$\sigma = \frac{4\pi W^4}{\lambda^2}\left[\frac{\sin(kW\sin\theta)}{kW\sin\theta}\right]^2$$

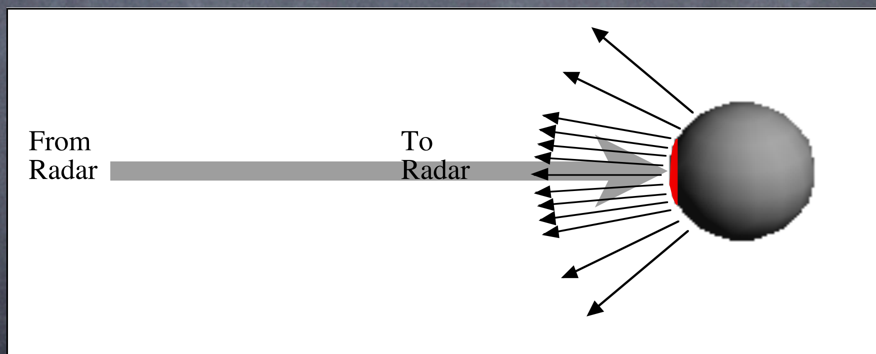
and for a circular plate of radius a

$$\sigma = \frac{\pi a^2}{\tan^2\theta}\left[J_1(2ka\sin\theta)\right]^2$$

where $k = 2\pi/\lambda$ and $J_1(\cdot)$ is the *first-order Bessel function of the first kind.*

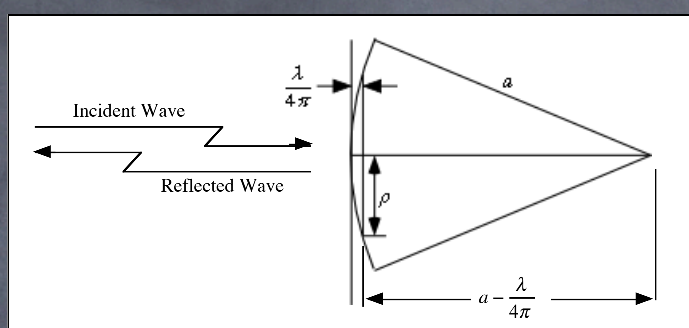
Back to the Sphere

In the optical region, most of the return comes from the "cap" of the sphere perpendicular to the direction of propagation.



Why is the RCS given by the geometric cross section? The "active area" is smaller.

Suppose we approximate the "cap" as a flat plate



How big is the cap?

Assume maximum deviation in "cap" is $\delta x = \lambda/4\pi$.

Then the radius ρ of the "cap" satisfies:

$$\rho^2 + \left(a - \frac{\lambda}{4\pi}\right)^2 = a^2$$

5.6

$$\rho^2 + \left(a - \frac{\lambda}{4\pi}\right)^2 = a^2$$

Solving for ρ yields

$$\begin{aligned}
\rho &= \sqrt{a^2 - \left(a - \frac{\lambda}{4\pi}\right)^2} \\
&= \sqrt{a^2 - \left(a^2 - \frac{a\lambda}{2\pi} + \frac{\lambda^2}{16\pi^2}\right)} \\
&= \sqrt{\frac{a\lambda}{2\pi} - \frac{\lambda^2}{16\pi^2}} \\
&\approx \sqrt{\frac{a\lambda}{2\pi}}, \qquad \text{for } a \gg \lambda.
\end{aligned}$$

5.7

Thus the area A_p of the "cap" is

$$A_p = \pi\rho^2 = \pi\left(\frac{a\lambda}{2\pi}\right) = \frac{a\lambda}{2}$$

The resulting RCS of this circular "plate" is

$$\sigma = \frac{4\pi A_p^2}{\lambda^2} = \frac{4\pi(a\lambda/2)^2}{\lambda^2} = \pi a^2.$$

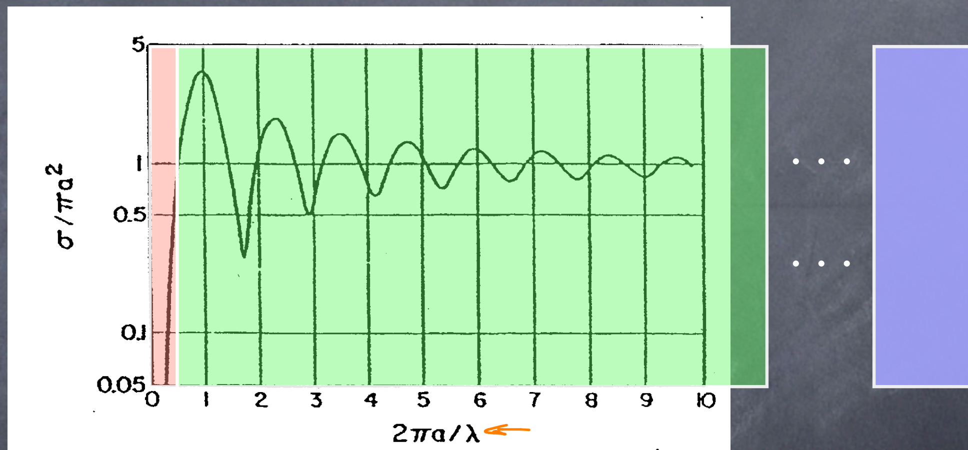
Cheating? For general δx:

$$\sigma = 16\pi^3\left(\frac{\delta x}{\lambda}\right)^2 a^2 \quad = \pi a^2\left[4\left(\frac{2\pi}{\lambda}\delta x\right)^2\right]$$

5.8

RCS of a (Perfectly Conducting) Sphere

A strong function of wavelength



The *Rayleigh Region*, where $2\pi a/\lambda < 0.4$

The *Mie Region*, where $0.4 < 2\pi a/\lambda < 20$

The *Optical Region*, where $2\pi a/\lambda > 20$

5.9

In the Rayleigh Region

($\lambda \gg$ maximum dimension)

Assume a smooth (small) object has volume V:

$$\sigma \approx \frac{4V^2}{\pi}\left(\frac{2\pi}{\lambda}\right)^4 = \frac{4k^4V^2}{\pi}$$

This is for an arbitrary object—not just sphere.

This is not quite right for a sphere:

$$\text{coefficient } \frac{64}{9} \approx 7 \neq 9$$

Recall for sphere: $\sigma \approx 9\pi a^2 \left(\frac{2\pi a}{\lambda}\right)^4 = \pi a^2 \left[9(ka)^4\right]$

Ruck et. al, Radar Cross Section Handbook, Plenum 1970.

Radar Return From Random Scatterers

Consider a radar illuminating a large number of random scatterers.

The received electric field due to the j-th scatterer is

$$E_j = A_j e^{i\phi_j}$$

where

$A_j =$ the magnitude of the field from the j-th scatterer

$\phi_j =$ phase of the field from the j-th scatterer

Assume the collection of scatterers have the following properties:

1. The size of any individual scatterer is independent of all others.

2. The locations of the scatterers are independent of each other.

3. Scatterer size is independent of position.

n.b. This would be satisfied by a nonhomogeneous marked Poisson Process.

For this situation, we can model $\{A_1, A_2, \ldots, A_n\}$ and $\{\phi_1, \phi_2, \ldots, \phi_n\}$ as jointly distributed random variables with the following properties:

1. $A_1, \ldots, A_n$ are independent.
2. $\phi_1, \phi_2, \ldots, \phi_n$ are independent.
3. $A_1, \ldots, A_n, \phi_1, \ldots, \phi_n$ are independent.

Note that properties 1 and 2 follow from property 3. We list separately for physical intuition.

We further assume:

4. $A_1, \ldots, A_N$ are *independent, identically distributed* (i.i.d.) random variables with mean μ_A and variance σ_A^2.
5. $\phi_1, \ldots, \phi_n$ are i.i.d. random variables uniformly distributed on the interval $[0, 2\pi)$.

These assumptions are reasonable if:

(a) We have no reason to believe any particular scatterer should be larger than any other.

(b) Scatterers are randomly distributed over a region many wavelengths long.

5.14

The total electric field at the receive antenna is the superposition of the individual E_j:

$$E = \sum_{j=1}^{n} E_j = \sum_{j=1}^{n} A_j e^{i\phi_j} = V e^{i\theta} = X + iY,$$

where

$$X = V\cos\theta = \sum_{j=1}^{n} A_j \cos\phi_j = \sum_{j=1}^{n} X_j$$

$$Y = V\sin\theta = \sum_{j=1}^{n} A_j \sin\phi_j = \sum_{j=1}^{n} Y_j,$$

and

$$X_j = A_j \cos\phi_j$$

$$Y_j = A_j \sin\phi_j$$

5.15

Note that

$$\mathrm{E}[X_j] = \mathrm{E}[A_j \cdot \cos\phi_j] = \mathrm{E}[A_j] \cdot \mathrm{E}[\cos\phi_j] = \mathrm{E}[A_j] \cdot 0 = 0$$

$$\mathrm{E}[Y_j] = \mathrm{E}[A_j \cdot \sin\phi_j] = \mathrm{E}[A_j] \cdot \mathrm{E}[\sin\phi_j] = \mathrm{E}[A_j] \cdot 0 = 0$$

and

$$\mathrm{var}[X_j] = \mathrm{E}[X_j^2] = \mathrm{E}[A_j^2] \cdot \mathrm{E}[\cos^2\phi_j] = (\mu_A^2 + \sigma_A^2) \cdot \frac{1}{2}$$

$$= \frac{\mu_A^2 + \sigma_A^2}{2}$$

and similarly

$$\mathrm{var}[Y_j] = \mathrm{E}[Y_j^2] = \cdots = (\mu_A^2 + \sigma_A^2) \cdot \frac{1}{2} = \frac{\mu_A^2 + \sigma_A^2}{2}$$

$$j = 1, \ldots, n$$

5.16

By the **Central Limit Theorem**,

$$X = \sum_{j=1}^{n} X_j \qquad \text{and} \qquad Y = \sum_{j=1}^{n} Y_j$$

are asymptotically Gaussian as $n \to \infty$.

Actually, $n \approx 15$–20 is pretty good.

Furthermore

$$\mathrm{E}[XY] = \mathrm{E}\left[\left(\sum_{j=1}^{n} X_j\right)\left(\sum_{k=1}^{n} Y_k\right)\right]$$

$$= \sum_{j=1}^{n}\sum_{k=1}^{n} \mathrm{E}[A_j A_k] \cdot \mathrm{E}[\cos\phi_j \sin\phi_k] = 0$$

Because $\mathrm{E}[\cos\phi_j \sin\phi_k] = 0$, for all $j, k = 1, \ldots, n$.

5.17

So

$$\mathrm{E}[XY] = \mathrm{E}[X] \cdot \mathrm{E}[Y]$$

$\Longrightarrow$ X and Y are *uncorrelated*

Recall, X and Y are also Gaussian

Gaussian AND Uncorrelated $\Rightarrow$ **Independent!**

X and Y are independent jointly-distributed Gaussians

$$\mathrm{E}[X] = \mathrm{E}[Y] = 0$$

$$\mathrm{var}[X] = \mathrm{var}[Y] = \sigma^2 = n(\mu_A^2 + \sigma_A^2)/2$$

5.18

So we have

$$Ve^{i\Theta} = X + iY,$$

where

$$V = \sqrt{X^2 + Y^2},$$

$$\Theta = \arctan(Y, X) \quad \text{(four-quadrant arctan)}$$

Define the received power as

$$P = V^2$$

To find $f_P(p)$, compute

$$f_{P\Theta}(p,\theta) = f_{XY}(x,y)\left|\frac{\partial(x,y)}{\partial(p,\theta)}\right|$$

and then

$$f_P(p) = \int_{-\infty}^{\infty} f_{P\Theta}(p,\theta)\, d\theta.$$

5.19

$$f_{P\Theta}(p,\theta) = f_{XY}(x,y)\left|\frac{\partial(x,y)}{\partial(p,\theta)}\right|$$

Now

$$f_{XY}(x,y) = f_X(x)f_Y(y) = \frac{1}{2\pi\sigma^2}\exp\left\{-\frac{(x^2+y^2)}{2\sigma^2}\right\}$$

$$x(p,\theta) = \sqrt{p}\cdot\cos\theta$$

$$y(p,\theta) = \sqrt{p}\cdot\sin\theta$$

$$\frac{\partial(x,y)}{\partial(p,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial p} & \frac{\partial y}{\partial p} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \frac{1}{2}p^{-1/2}\cos\theta & \frac{1}{2}p^{-1/2}\sin\theta \\ -p^{1/2}\sin\theta & p^{1/2}\cos\theta \end{vmatrix} = \frac{1}{2}\left(\cos^2\theta + \sin^2\theta\right) = \frac{1}{2}$$

So

$$f_{P\Theta}(p,\theta) = \frac{1}{2\pi\sigma^2}\exp\left\{-\frac{(p\cos^2\theta + p\sin^2\theta)}{2\sigma^2}\right\}\cdot 1_{[0,\infty)}(p)\cdot 1_{[0,2\pi)}(\theta)\cdot\left|\frac{1}{2}\right|$$

$$= \frac{1}{4\pi\sigma^2}\exp\left\{-\frac{p}{2\sigma^2}\right\}\cdot 1_{[0,\infty)}(p)\cdot 1_{[0,2\pi)}(\theta)$$

5.20

$$f_{P\Theta}(p,\theta) = \frac{1}{4\pi\sigma^2}\exp\left\{-\frac{p}{2\sigma^2}\right\}\cdot 1_{[0,\infty)}(p)\cdot 1_{[0,2\pi)}(\theta)$$

It follows that

$$f_P(p) = \int_{-\infty}^{\infty} f_{P\Theta}(p,\theta)\,d\theta = \boxed{\frac{1}{2\sigma^2}\exp\left\{-\frac{p}{2\sigma^2}\right\}\cdot 1_{[0,\infty)}(p)}$$

An exponential pdf with mean $2\sigma^2$

5.21

It can also be shown that

$$f_\Theta(\theta) = \int_{-\infty}^{\infty} f_{P\Theta}(p,\theta)\,dp = \boxed{\frac{1}{2\pi}\cdot 1_{[0,2\pi)}(\theta)}$$

Uniform distribution on $[0,2\pi)$

Summarizing, we have the well known results

$$f_P(p) = \frac{1}{2\sigma^2}\exp\left\{-\frac{p}{2\sigma^2}\right\}\cdot 1_{[0,\infty)}(p),$$

$$f_\Theta(\theta) = \frac{1}{2\pi}\cdot 1_{[0,2\pi)}(\theta).$$

Sometimes, in addition to the complex random field return $Ve^{i\Theta}$, there is a constant component a as well:

$$W = |Ve^{i\Theta} + ae^{i\Psi}|$$

Resultant Magnitude — Random Component — Constant Component

a = a non-random constant,

$\Psi =$ a random variable uniformly distributed on $[0, 2\pi)$.

It can be shown that the amplitude W has pdf

$$f_W(w) = \frac{w}{\sigma^2} \exp\left\{-\frac{(w^2+a^2)}{2\sigma^2}\right\} I_0\left(\frac{wa}{\sigma^2}\right) \cdot 1_{[0,\infty)}(w)$$

Rician Amplitude Distribution

Received Power: $P = V^2$

It can be shown that

$$f_P(p) = \left(\frac{1+m^2}{\mu_p}\right) e^{-m^2} \exp\left\{-p\left(\frac{1+m^2}{\mu_p}\right)\right\} I_0\left(2m\sqrt{\frac{p(1+m^2)}{\mu_p}}\right) \cdot 1_{[0,\infty)}(p),$$

Rician Power Distribution

where

$$m = \sqrt{\frac{a^2}{2\sigma^2}}$$

and

$$I_0(\cdot) = \textit{modified Bessel function of order zero.}$$

RCS of Corner Reflectors

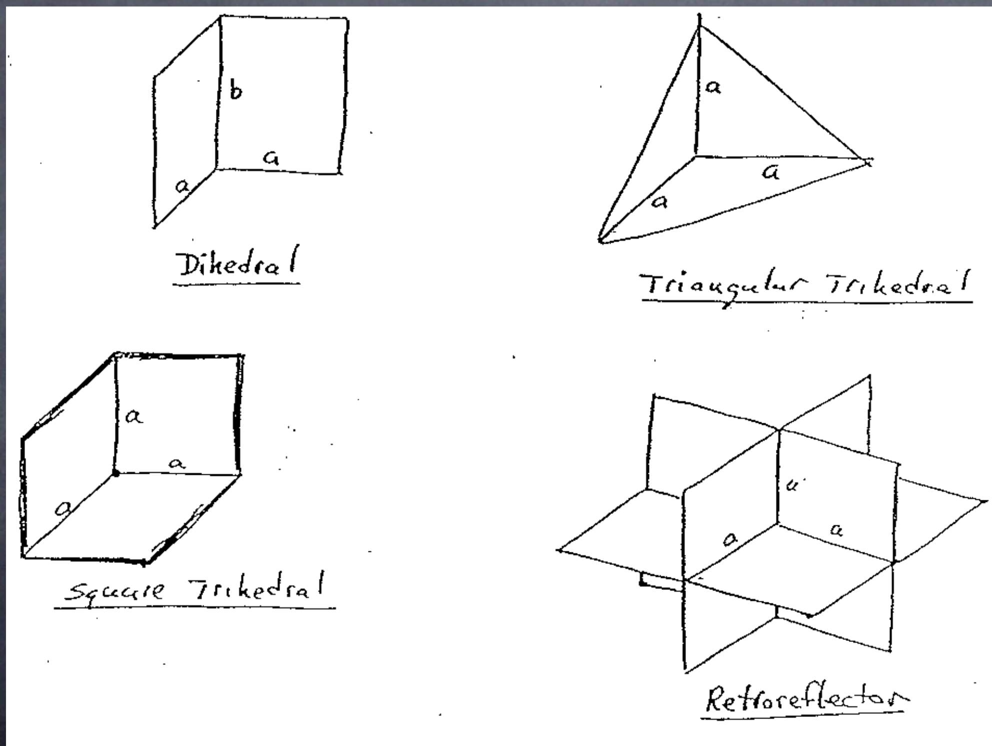
- A corner reflector is a radar target constructed by letting a number of planar surfaces come together, forming a corner.
- Corner reflectors tend to have the property that they reflect strongly back in the direction of the incident wave.
- For this reason, they are also called retro-reflectors.
- One common corner reflector is the "corner cube," used to construct optical bicycle reflectors.

Another Recreational Use...

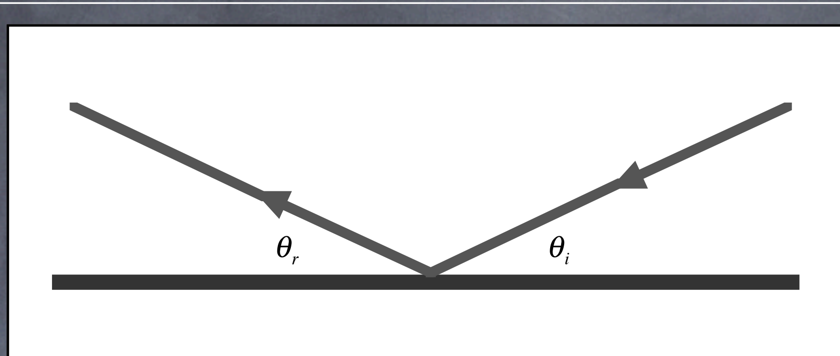
A typical sailboat corner reflector
(hung from mast)

A typical situation where they are useful!

Common Corner Reflector Geometries



Fresnel's Law of Reflection



"The angle of incidence equals the angle of reflection."

θ_i = Angle of Incidence

θ_r = Angle of Reflection

n.b., We assume surface is large compared to a wavelength (Ray Optics Approximation)