Recall... The Friis Equation

Suppose we have two antennas “pointing at each other” a large distance $R$ apart.

If

$P_T = \text{transmitted power}$

$P_R = \text{received power}$

then

$$\frac{P_R}{P_T} = \frac{A_T A_R}{\lambda^2 R^2}.$$  \hspace{1cm} \text{(Friis Equation)}
Antenna Gain over an Isotropic Radiator

- **Isotropic Radiator**: An antenna that radiates energy uniformly in all directions (transmit).

- On receive, it is equally sensitive to energy from all directions (by reciprocity).

If at a distance $R$ from an isotropic radiator, we place a receive aperture $A_R$

$$
\frac{P_R}{P_T} = \frac{A_R}{4\pi R^2}
$$

$A_R = \frac{\lambda^2}{4\pi R^2} \Rightarrow A_i = \frac{\lambda^2}{4\pi}$

Note that the usual requirements for aperture size are not met.

However, it works in the Friis equation.

For this reason we will find it useful.
The gain of $A_T$ over $A_i$ is

$$G = \frac{(P_R/P_T)_T}{(P_R/P_T)_i} = \frac{(P_R)_T}{(P_R)_i} = \frac{4\pi A_T}{\lambda^2}$$

So in general, the relation between the effective area $A$ and gain $G$ of antenna is

$$G = \frac{4\pi A}{\lambda^2} \quad A = \frac{\lambda^2 G}{4\pi}$$

Gain is often expressed in dB:

$$G(\text{dB}) = 10 \log_{10} G \quad (\text{dB})$$

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**Antenna Gain...**

- By reciprocity, the gain of an antenna on transmit is equal to the gain of an antenna on receive.

- The Friis Equation can be written in terms of antenna gains:

$$\frac{P_R}{P_T} = \frac{G_T G_R \lambda^2}{16\pi^2 R^2}$$
Antenna Directivity and Beam Pattern

For a uniformly illuminated aperture

\[ G_T = \frac{4\pi A_{TG}}{\lambda^2} \]

If \( A_{TG} \) is not uniformly illuminated

\[ G_T = \frac{4\pi A_T}{\lambda^2} = \frac{4\pi A_{TG}}{\lambda^2} \eta \]

If we go off axis by \( \theta \) (azimuth) and \( \phi \) (elevation)
the gain is not as large as in direction \((\theta, \phi) = (0, 0)\).
It can be written as

\[ G(\theta, \phi) = \frac{4\pi A_{TG}}{\lambda^2} \cdot f(\theta, \phi) \]

\[ f(\theta, \phi) = \left( \frac{1}{A_{TG}} \right) \left| \frac{1}{\int \int_{A_{TG}} \exp \left\{ i \frac{2\pi}{\lambda} [x \sin \theta + y \sin \phi] \right\} E(x, y) \, dx \, dy} \right|^2 \frac{\int \int_{A_{TG}} |E(x, y)|^2 \, dx \, dy}{\int \int_{A_{TG}} |E(x, y)|^2 \, dx \, dy} \cdot \]

Looking back at the aperture off-axis we see a "virtual aperture" with a phase shift across it.

In general, at angle \((\theta, \phi)\) we have

\[ G(\theta, \phi) = \frac{4\pi A_{TG}}{\lambda^2} \cdot f(\theta, \phi) \]
\[ f(\theta, \phi) = \left( \frac{1}{A_{TG}} \right) \left| \frac{\int \int_{A_{TG}} \exp \left\{ \frac{2\pi}{\lambda} \left[ x \sin \theta + y \sin \phi \right] \right\} E(x, y) \, dx \, dy}{\int \int_{A_{TG}} |E(x, y)|^2 \, dx \, dy} \right|^2 \]

This expression acts like an efficiency, taking on values between 0 and 1.

It provides a measure of the directivity of the antenna for transmitting and receiving power.

We call this quantity the beam pattern of the antenna.

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**The Radar Equation**

Given that power \( P_T \) is transmitted, what is the received power \( P_R \)?

To answer this, we must understand the target’s behavior.
Assume target has following characteristics:

(i) As a receive aperture, it has $A_R = \sigma \text{ (m}^2\text{)}$;
(ii) It reradiates all of this received energy isotropically.
The power received by the target is given by

\[ \frac{P_\sigma}{P_T} = \frac{A \sigma}{\lambda^2 R^2} \]

The fraction of the reradiated power received is

\[ \frac{P_R}{P_\sigma} = \frac{A_i A}{\lambda^2 R^2} = \frac{(\lambda^2/4\pi) A}{\lambda^2 R^2} = \frac{A}{4\pi R^2} \]

It follows that

\[ \frac{P_R}{P_T} = \frac{P_R}{P_\sigma} \cdot \frac{P_\sigma}{P_T} = \frac{A^2 \sigma}{4\pi \lambda^2 R^4} \]

The Radar Equation

\[ \frac{P_R}{P_T} = \frac{A^2 \sigma}{4\pi \lambda^2 R^4} \]

(i) \( \frac{P_R}{P_T} \) proportional to \( \frac{1}{R^4} \)

(ii) \( \frac{P_R}{P_T} \) proportional to \( \sigma \)
Notes on Radar Equation

Alternative form using antenna gain(s) instead of effective area can be derived.

Bistatic version with different transmit and receive ranges and effective areas can be derived (requires generalization of radar cross section.)

As we will see, radar cross section and geometric cross section can be quite different—don’t let this throw you for now.

Radar Targets

\[
\frac{P_R}{P_T} = \frac{A^2 \sigma}{4\pi \lambda^2 R^4}
\]

RCS is used to characterize the scattering characteristics of target.

Defined in terms of hypothetical target—defines an equivalence class of targets.

Is used to describe physical targets that behave nothing like the hypothetical target that defines it. This is OK!
RCS Contributing Factors

- Size of Object
- Shape of Object
- Wavelength of Radiation
- Material(s) Object is Made of
- Orientation w.r.t. Radar

<table>
<thead>
<tr>
<th>Object</th>
<th>RCS (m²)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Small Insect (fly)</td>
<td>$10^{-5}$</td>
</tr>
<tr>
<td>Large Insect (locust)</td>
<td>$10^{-4}$</td>
</tr>
<tr>
<td>Medium-Sized Bird</td>
<td>0.001</td>
</tr>
<tr>
<td>Large Bird</td>
<td>0.01</td>
</tr>
<tr>
<td>Small Open Boat</td>
<td>0.02</td>
</tr>
<tr>
<td>Small Missile</td>
<td>0.1</td>
</tr>
<tr>
<td>Man</td>
<td>1</td>
</tr>
<tr>
<td>Small Single-engine Airplane</td>
<td>1</td>
</tr>
<tr>
<td>Small Fighter or Four-Passenger Jet</td>
<td>2</td>
</tr>
<tr>
<td>Helicopter</td>
<td>2</td>
</tr>
<tr>
<td>Bicycle</td>
<td>2</td>
</tr>
<tr>
<td>Small Pleasure Boat (20-30 ft.)</td>
<td>2</td>
</tr>
<tr>
<td>Large Tactical Fighter Airplane</td>
<td>6</td>
</tr>
<tr>
<td>Cabin Cruiser (40-50 ft.)</td>
<td>10</td>
</tr>
<tr>
<td>Large Bomber or Commercial Airliner</td>
<td>40</td>
</tr>
<tr>
<td>Jumbo Jet</td>
<td>100</td>
</tr>
<tr>
<td>Automobile</td>
<td>100</td>
</tr>
<tr>
<td>Pickup Truck</td>
<td>200</td>
</tr>
<tr>
<td>Ship</td>
<td>3000–1000000</td>
</tr>
</tbody>
</table>
For real targets, we almost never know the value of the RCS a priori.

We may know a range of values that $\sigma$ may lie in:

$$\sigma_L \leq \sigma \leq \sigma_U$$

Sometimes it makes sense to treat $\sigma$ as a random variable:

$$\sigma(\omega) \text{ defined on } (\mathcal{S}, \mathcal{F}, \mathbb{P})$$

Sometimes it makes sense to treat $\sigma$ as a random process:

$$\sigma(t, \omega) \text{ defined on } (\mathcal{S}, \mathcal{F}, \mathbb{P})$$

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**RCS of a Sphere**

A strong function of wavelength

The *Optical Region*, where $2\pi a/\lambda > 20$
RCS of a (Perfectly Conducting) Sphere

A strong function of wavelength

The Rayleigh Region, where $2\pi a/\lambda < 0.4$

The Mie Region, where $0.4 < 2\pi a/\lambda < 20$

The Optical Region, where $2\pi a/\lambda > 20$

In the Optical Region, $\sigma \approx \pi a^2$. This is the geometric cross section of a sphere.

Because a sphere is invariant to changes in its orientation, it makes a convenient calibration target.

In the Rayleigh Region, where $\lambda >> a$,

$$\sigma \approx 9\pi a^2 \left(\frac{2\pi a}{\lambda}\right)^4 = \pi a^2 \left[9(ka)^4\right]$$
RCS of a (Perfectly Conducting) Sphere

In the Mie Region, where $\lambda \approx a$, “creeping waves” travel around the sphere and interfere with the specular reflection:

This gives rise to the “resonance” seen in this region.

RCS of a Perfectly Conducting Plate

Consider a perfectly conducting plate with dimensions much greater than $\lambda$. Assume area $A_p$.

Assume perpendicular orientation to incident wave.

In the far field, the plate is uniformly illuminated.

The plate reflects or radiates the wave as if it were a uniformly illuminated aperture of area $A_p$. (It is!)
Applying the Friis equation twice—once for each trip—we get

\[
\frac{P_R}{P_T} = \frac{P_\sigma}{P_T} \cdot \frac{P_R}{P_\sigma} = \frac{AA_p}{\lambda^2 R^2} \cdot \frac{A_p A}{\lambda^2 R^2} = \frac{A^2 A^2_p}{\lambda^4 R^4}
\]

But

\[
\frac{P_R}{P_T} = \frac{A^2 \sigma}{4 \pi \lambda^2 R^4}
\]

Equating these expressions, we have

\[
\frac{A^2 A^2_p}{\lambda^4 R^4} = \frac{A^2 \sigma}{4 \pi \lambda^2 R^4} \quad \Rightarrow \quad \sigma = \frac{4 \pi A^2_p}{\lambda^2}.
\]